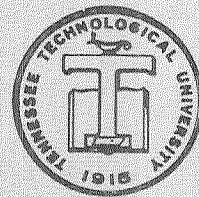


ENGINEERING RESEARCH SERIES

N71-30356

NASA CR-115074



IDENTIFICATION OF A LINEAR SYSTEM FROM
SAMPLED NOISY DATA

by

Dennis L. Luckinbill, Project Director
Venilal H. Sumaria

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Engineering Complex

TENNESSEE TECHNOLOGICAL UNIVERSITY
COOKEVILLE, TENNESSEE

CR-115074

FINAL PROJECT REPORT
Part C

IDENTIFICATION OF A LINEAR SYSTEM FROM
SAMPLED NOISY DATA

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Contract No. NAS 9-10385

NASA Manned Spacecraft Center
Houston, Texas

June 1971

ACKNOWLEDGEMENTS

This paper is based upon the masters thesis of the second author.

This research was supported by NASA Contract No. NAS 9-10385.

The first and second author were both supported by this NASA Contract.

The support is gratefully acknowledged.

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NOMENCLATURE

a	An arbitrary constant
a_i	Parameters to be identified
b	An arbitrary constant
b_i	Parameters to be identified
c	An arbitrary constant
k	k th iteration
t	Time in seconds
A	Coefficient matrix
B	Input Vector
G	Coefficient matrix
L	Least square criterion
S	Input matrix
T	The average time in seconds
x	Variable
y	Variable
y_i	State variables

Greek Symbols

α	Initial condition vector
β	Convergence factor
ρ	Perturbation factor
η	Alteration in initial condition
ω_n	Natural frequency in radians per seconds
ξ	Damping ratio

ξ_i ith data set
 ϕ An angle
 γ Vector of combining coefficients
 $\Phi_p(t_0)$ Perturbed initial condition matrix

General

\dot{x} First derivative of x with respect to time
 $f(x)$ Function of x
 $F(x)$ Function of vector x
 $\frac{\partial F}{\partial \bar{x}}$ Partial derivative of F with respect to vector x
 $\frac{d^n x}{dt^n}$ Nth derivative of x with respect to time
 \subset Subset of
 H.O.T. Higher order terms

CHAPTER I

INTRODUCTION

Parameter identification is one of the important phases of automatic control systems. Many control processes are directly or indirectly related to some kind of identification of a process model. For example, adaptive control, optimal control, and recent interest in biological systems require identification of parameters to create better mathematical models to represent the physical system in an approximate form.

Creating linear models for nonlinear systems is a very important aspect of system analysis. Linear systems are easier to work with and identification of nonlinear systems is more complex since their response characteristics are amplitude as well as frequency dependent, Eveleigh (1967).

Many identification techniques are available in the mathematical literature. Least square estimation is one of the oldest known methods. A recently developed techniques called quasilinearization and method of perturbations have a great future for application to linear or nonlinear multivariable systems. Quasilinearization or method of perturbations can be easily implemented on digital computers which increases their importance for application as on line identification methods.

I. STATEMENT OF THE PROBLEM

The problem is to identify a best approximate linear system with a step input when sampled noisy data from the output of a nonlinear

system are given. A modified quasilinearization parameter identification technique known as the method of perturbations is the method used for finding the best approximation.

The data obtained for the numerical experiments in this problem are the result of simulation of the control system for automated environmental control of an acoustic test facility at NASA, Manned Spacecraft Center, Houston, Texas, Luckinbill (1970). This simulation which is considered as a physical system for this problem is shown in Figure 1. The physical system that is considered in Figure 1 includes a R.M.S. detector. The output of the R.M.S. detector provides the data for this problem. The simulated system includes a sixth order system with the coefficients as shown in Figure 1. The R.M.S. detector used is described as follows:

$$R(t) = \left[\frac{1}{t-(t-T)u(t-T)} \int_{(t-T)u(t-T)}^t x^2(\xi) d\xi \right]^{1/2}$$

where

$u(t-T)$ = unit step function
 T = averaging time
 $x(t)$ = output of the sixth order simulated system.

II. PREVIOUS WORK

Approximating nonlinear systems by linear models has created a growing interest in recent years. Parameter identification plays an important part in obtaining linear models. Many methods of identification

exist in the literature.

Many of the present identification techniques are summarized by Eveleigh (1967) and Sage and Cuenod (1967). Eveleigh (1967) has described the relationship of identification and adaptive control problems. Least square estimation of parameters was developed separately by Gauss and Legendre in early 1800's to estimate the parameters of the motion of heavenly bodies using physical and astronomical data, Sorenson (1970). Sorenson also summarizes the development of least square estimation and Kalman filter theory.

Lee (1967) roughly divides all adaptive control system techniques into two classes: those using explicit identification methods and those using nonidentification methods. Nahi (1969) has described identification by maximum likelihood and steepest descent. Kovanic (1967) has considered the problem of finding some or all parameters of the unknown system on the basis of digitally treating noisy data and has shown the method of least pseudosquares to yield efficient results.

Quasilinearization is a recently developed and very efficient technique for identification of linear and nonlinear systems parameters. Bellman formulated the basic ideas behind quasilinearization also called the Newton-Raphson-Kontorovick expansion in function space. Kalaba in 1959 added a great deal of mathematical rigor necessary for the successful employment of this method, Bellman and Kalaba (1965). Holloway (1968) used this method to investigate the possibility of identification of parameters for describing the earth's geopotential from synchronous satellite data. Paine (1967) reviews the use of

this method in the computation of optimal control. Lee (1968) has considered the solution of nonlinear ordinary differential equations with nonlinear boundary conditions.

Luckinbill and Childs (1968) have applied the method of perturbations to the identification of parameters in partial differential equations. Smith (1969) has considered the method of perturbations and steepest descent for system identification using a hybrid computing facility in order to compare their suitability as on-line identification methods. Duval (1969) has also applied the method of perturbations to an adaptive control problem using a hybrid computer.

CHAPTER II

THEORY OF IDENTIFICATION

I. REVIEW OF QUASILINEARIZATION METHOD

The basic concept of quasilinearization is small-signal linearization of system response about a nominal path through state space, Eveleigh (1967). This method requires the appropriate knowledge of the form of the differential equation. The differential equation must be reduced to a set of first order differential equations and coupled with the set of differential equations of the parameters. Coupling these sets of differential equations forms a new set of non-linear, first order differential equations. Experimental data may be used for the boundary conditions. The solution of this boundary value problem can be accomplished as follows: (1) the linearization of the nonlinear equations by Newton-Raphson-Kantorovich expansion formula which Bellman and Kalaba (1965) refer to as quasilinearization and (2) the solution of the remaining linearized equations by an iterative method on a digital computer.

To illustrate this method consider the differential equation

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_n x = a_{n+1} \quad (2.1)$$

where a_{n+1} is a step input.

This can be written as a set of first-order differential equations

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n, a_1, a_2, \dots, a_{n+1}) \quad i=1, 2, 3, \dots, n \quad (2.2)$$

$$\text{where } \dot{x}_i = \frac{dx_i}{dt} \quad i = 1, 2, \dots, n \quad (2.3)$$

$$\text{Let } x_{n+j} = a_j \quad j = 1, 2, \dots, n+1 \quad (2.4)$$

The set of differential equations for the unknown constants will be

$$\dot{a}_j = \dot{x}_{n+j} = \sigma \quad j = 1, 2, \dots, n+1 \quad (2.5)$$

Rewriting equation (2.2)

$$\dot{x}_j = f_j(x_1, \dots, x_{2n+1}) \quad j = 1, \dots, 2n+1 \quad (2.6)$$

Equation (2.6) may be represented in matrix form as

$$\dot{\bar{X}} = F(\bar{X}) \quad (2.7)$$

Using Newton-Raphson-Kantorovich expansion formula one can linearize equation (2.7) as

$$\dot{\bar{X}}_{k+1} = F(\bar{X}_k) + \frac{\partial F}{\partial \bar{X}_k} (\bar{X}_{k+1} - \bar{X}_k) + \text{H.O.T.} \quad (2.8)$$

where $k = k\text{th iteration}$

H.O.T. = higher order terms

$\frac{\partial F}{\partial \bar{X}_k}$ = the Jacobian matrix

$$\frac{\partial F}{\partial \bar{X}_k} = \begin{bmatrix} \frac{\partial f_1}{\partial x_{1,k}} & \dots & \frac{\partial f_1}{\partial x_{n,k}} \\ \vdots & & \vdots \\ \frac{\partial f_{2n+1}}{\partial x_{1,k}} & \dots & \frac{\partial f_{2n+1}}{\partial x_{n,k}} \end{bmatrix} \quad (2.9)$$

The iterative solution of equation (2.8) requires a vector $\bar{X}_k(t)$. This vector is generated by using the best estimate of $\bar{X}(0)$ as initial conditions for solving equation (2.7) as the k th solution. Equation (2.8) may be written as

$$\dot{\bar{X}}_{k+1} = G \bar{X}_{k+1} + \bar{B} \quad (2.10)$$

where G and B are functions of $\bar{X}_k(t)$

Equation (2.10) can be solved by superposition of particular solutions (see Appendix B). This method is explained in detail by Luckinbill and Childs (1968) who point out that the use of particular solutions has the feature of giving an explicit indication of the degree of convergence of the iterative process.

II. AN EXAMPLE

To illustrate the method of identification by the method of perturbations, consider the second order model which is to be used for identification for the problem

$$\ddot{y} + a_1 \dot{y} + a_2 y = a_3 \quad (2.11)$$

In equation (2.11) a_1 , a_2 , and a_3 are unknown constants, a_3 is the step input to the system and a_1 , a_2 , and a_3 are to be identified.

Rewriting equation (2.11)

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -y_3 y_2 - y_4 y_1 + y_5 \end{aligned} \quad (2.12)$$

where

$$y = y_1$$

$$\dot{y} = y_2$$

$$a_1 = y_3$$

$$a_2 = y_4$$

$$a_3 = y_5$$

y_3 , y_4 , and y_5 are considered to vary slowly enough with time over the identification period to be assumed constant

$$\dot{y}_3 = \dot{y}_4 = \dot{y}_5 = 0 \quad (2.13)$$

Coupling equations (2.12) and (2.13) and rewriting in matrix form

$$\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -y_4 & -y_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \quad (2.14)$$

Applying the Newton-Raphson-Kantorovich expansion, equation (2.8), and dropping the higher order terms

$$\dot{\bar{y}}_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -(y_4)_k & -(y_3)_k & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bar{y}_{k+1} + \frac{\partial F}{\partial \bar{y}_k} \begin{bmatrix} (y_1)_{k+1} - (y_1)_k \\ (y_2)_{k+1} - (y_2)_k \\ (y_3)_{k+1} - (y_3)_k \\ (y_4)_{k+1} - (y_4)_k \\ (y_5)_{k+1} - (y_5)_k \end{bmatrix} \quad (2.15)$$

where

$$\frac{\partial F}{\partial \bar{y}_k} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -(y_4)_k & -(y_3)_k & -(y_2)_k & -(y_1)_k & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.16)$$

and k is the k th iteration.

Equation (2.15) can be written as

$$\dot{\bar{y}}_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -(y_4)_k & -(y_3)_k & -(y_2)_k & -(y_1)_k & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bar{y}_{k+1} + \begin{bmatrix} 0 \\ 2(y_4)_k (y_1)_k + 2(y_3)_k (y_2)_k + (y_5)_k \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.17)$$

The resulting linear boundary value problem can be solved by the method of superposition (see appendix B).

The solution of equation (2.17) can be written in the form of equation (B.2) as the superposition of particular solutions

$$\bar{y}_{k+1} = \Phi_p(t) \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{bmatrix} \quad (2.18)$$

The initial condition matrix used to generate $\Phi_p(t)$ as given by equation (B.11) will be

$$\Phi_p(t_0) = \begin{bmatrix} \alpha_1^{\rho_1} & \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2^{\rho_2} & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3^{\rho_3} & \alpha_3 & \alpha_3 & \alpha_3 \\ \alpha_4 & \alpha_4 & \alpha_4 & \alpha_4^{\rho_4} & \alpha_4 & \alpha_4 \\ \alpha_5 & \alpha_5 & \alpha_5 & \alpha_5 & \alpha_5^{\rho_5} & \alpha_5 \end{bmatrix} \quad (2.19)$$

If it is assumed that the two initial conditions $y(0)$ and $\dot{y}(0)$ are known, γ_1 and γ_2 will be zero so that only four particular solutions are required and equation (2.19) reduces to

$$\Phi_p(t_0) = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3^{\rho_3} & \alpha_3 & \alpha_3 & \alpha_3 \\ \alpha_4 & \alpha_4^{\rho_4} & \alpha_4 & \alpha_4 \\ \alpha_5 & \alpha_5 & \alpha_5^{\rho_5} & \alpha_5 \end{bmatrix} \quad (2.20)$$

Each column of $\Phi_p(t)$ must then satisfy equation (2.17). A set of linear algebraic equations will be generated from superposition of the independent solutions in $\Phi_p(t)$ which must satisfy the boundary conditions. In the case of this paper all are given on one element

of the state vector. This set of equations can be solved for \bar{y} which from equation (2.18) will yield a new set of initial conditions, $\bar{y}_{k+1}(0)$ from which to begin another iteration.

CHAPTER III

APPROXIMATE LINEAR MODEL OF A NONLINEAR SYSTEM

I. BACKGROUND OF DATA BEING FIT

The data considered for this problem is the output of the physical system explained in the statement of the problem. The digital RMS detector has an averaging time of $T = 1$ second after the first one-second time frame. For the first one-second time frame all the samples obtained from the output of the sixth order system are used in the RMS calculations.

Two sets of data are considered for the problem. The first set is generated by an input to the physical system which is a narrow band random noise with a 25.0 Hertz center frequency and a gain of 1 as shown in Figure 1. This digital output is approximated by the solid line in Figure 5. Output up to 3.5 seconds is considered for the problem. This output is considered to be stationary and ergodic. The second set of data is obtained as a special case of the narrowband random input. In this case sinusoidal input with 25 Hertz sine wave is used with a gain of 7.68698. This digital output is shown by a dashed line in the Figure 5. Continuous curves in Figure 5 are plotted from the output of a digital simulation whose integration interval is .004 second.

By comparing both sets of data, it can be seen from Figure 5 that the sinusoidal data has a faster rise time than the random data in the transient region. The sinusoidal data has a steady state value

of approximately 5.0555 whereas the random data response is randomly oscillating approximately around 5.0555.

II. LEAST SQUARE FITTED MODELS

This paper is primarily concerned with identification of parameters when a large amount of data is available on one of the state variables. This data together with the set of differential equations describes a boundary value problem. When only certain boundary conditions are given, one has to fit the model to "best" satisfy those boundary conditions. But if a large number of boundary conditions are given so close to each other that they could be represented by a continuous curve, a selection of some boundary conditions should be made which would describe the continuous curve in the same "best" manner. Table 1 shows the minimum number of boundary conditions necessary to identify a linear model with a steady state constraint and also with no such constraint on the parameters.

If we are to reduce the number of data points, a question must be raised as to a practical number. Two considerations must be made. First, that for a larger number of data points more multiplications and summations are involved and thus more numerical round off error. However, the second consideration states that if we neglect the numerical round off error, a larger number of data points will lead to a better fit of the data. To show this let ξ_n, ξ_{n-1}, \dots , and ξ_m be $(n-m+1)$ sets of data where ξ_m contains m data points which is the minimum number of boundary conditions required to solve the given differential equation.

TABLE I

MINIMUM NUMBER OF BOUNDARY CONDITIONS

REQUIRED FOR A MODEL

Order of model	No Constraints on Parameters to be identified	Steady state constraint
1	3	2
2	5	4
3	7	6
4	9	8

Thus the set ξ_k , where $m \leq k \leq n$, forms an overdetermined boundary value problem. ξ_n is defined such that it contains n elements where ξ_{n-1} is a subset of ξ_n or more fully ξ_n contains all the data points

$$\xi_m \subset \xi_{m+1} \subset \dots \subset \xi_n$$

Z_i available and is the largest set.

Let us define a least square criterion L

$$L_n = \min_{\bar{a}} \sum_{i=1}^n [Z_i - y_n(t_i)]^2$$

where $y_n(t_i)$ = a solution of the differential equation being identified using n boundary conditions.

\bar{a} = the vector of unknown parameters being identified within the specified differential equation.

Likewise,

$$L_{n-1} = \min_{\bar{a}} \left[\sum_{\substack{i=1 \\ j=i}}^n \left\{ Z_i - y_{n-1}(t_i) \right\}^2 \right] + \left\{ Z_j - y_{n-1}(t_i) \right\}^2$$

In the above equations, L_n is the least square error for a solution of the differential equation using n boundary conditions to identify the parameters.

Assuming that the trajectory y_n is an optimum, any other trajectory will have a larger least square error. Using a set of boundary values ξ_{n-1} , a trajectory y_{n-1} would be obtained such that

$$L_n \leq L_{n-1}$$

It can be shown in a similar manner that

$$L_n \leq L_{n-1} \leq \dots \leq L_m$$

Since numerical round-off error is difficult to evaluate an arbitrary decision was made to use 35 boundary values for the random data and 34 boundary values for the sinusoidal data. These boundary values and time of the boundary values are listed in Table 2.

Weighting factors for the boundary values is an important factor in least square analysis of data. However, for this problem, it has been considered that all of the errors of the boundary conditions will have equal importance so that the weighting factors are all equal.

Identified linear models are presented in Figures 6 through 13. In Figure 6 is shown an optimum linear first order solution for the random data described earlier in this chapter.

The differential equation used for the solution in Figure 6 is

$$\ddot{x} + a_1 \dot{x} = a_2 \quad (3.1a)$$

where

$$a_1 = 2.999819$$

$$a_2 = 15.16576$$

This linear model has a steady state constraint

$$a_2 = 5.0555 a_1$$

TABLE II

BOUNDARY CONDITIONS

Narrow band random noise input to the physical system			Sinusoidal input to the physical System	
No	Time in Seconds	Boundary Value	Time in Seconds	Boundary Value
1	0.1	0.299414	0.04	0.013744
2	0.2	1.277949	0.08	0.06468099
3	0.3	1.719279	0.12	1.462720
4	0.4	2.92758	0.24	3.661570
5	0.5	4.621070	0.36	4.348080
6	0.6	5.42318	0.48	4.50155
7	0.7	5.381490	0.60	4.619630
8	0.8	5.175819	0.72	4.696130
9	0.9	5.045070	0.84	4.748690
10	1.0	4.77436	0.96	4.788369
11	1.1	4.80693	1.08	5.004510
12	1.2	4.858879	1.20	5.120529
13	1.3	4.887460	1.32	5.047359
14	1.4	5.447889	1.44	5.053269
15	1.5	5.860999	1.56	5.057610
16	1.6	6.116050	1.68	5.055420
17	1.7	6.164220	1.80	5.055699
18	1.8	6.287990	1.92	5.055900
19	1.9	6.296189	2.04	5.055860
20	2.0	6.333739	2.16	5.055810
21	2.1	6.343800	2.28	5.055710
22	2.2	6.343829	2.40	5.055650
23	2.3	6.43925	2.54	5.055610
24	2.4	5.859400	2.76	5.055510
25	2.5	4.781469	2.88	5.055449
26	2.6	3.531750	3.00	5.055429
27	2.7	3.146099	3.12	5.055349
28	2.8	2.950950	3.24	5.055349
29	2.9	3.112769	3.36	5.055280
30	3.0	3.88759	3.48	5.055280
31	3.1	4.354159	3.60	5.05524
32	3.2	4.592560	3.72	5.055160
33	3.3	4.723069	3.84	5.055000
34	3.4	4.729449	3.96	5.055079
35	3.5	4.776139		

and the least square error is 38.13594. This first order model has three first order differential equations for the purpose of identification. The initial conditions $x(0)$ is considered to be known and equal to zero, and the parameter a_2 is linearly dependent on the parameter a_1 . Therefore, only the parameter a_1 is to be identified for this model. The initial condition matrix, equation (B.11), will be

$$\Phi_p(t_0) = \begin{bmatrix} \alpha_1 \rho_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \rho_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 \rho_3 & \alpha_3 \end{bmatrix} \quad (3.2)$$

However, since the initial condition $x(0)$ is known and a_2 is dependent on a_1 as described above, only two independent solutions are required. Thus, the initial condition matrix reduces to

$$\Phi_p(t_0) = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 \rho_2 & \alpha_2 \\ c \alpha_2 \rho_2 & c \alpha_2 \end{bmatrix} \quad (3.3)$$

where

$$\begin{aligned} \rho_2 &= 1.2 \\ c &= 5.0555 \end{aligned}$$

The new initial condition vector is given by equation (B.12) as,

$$\bar{x}(t_0) = \Phi_p(t_0) \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \quad (3.4)$$

where γ_1 and γ_2 were found to be -1.44689×10^{-5} , and 1.000074 respectively at convergence of the iteration scheme.

In Figure 10 is shown an optimum linear first-order solution of the sinusoidal data as described earlier in this chapter where

$$\dot{x} + a_1 x = a_2 \quad (3.1b)$$

and

$$a_1 = 3.824817$$

$$a_2 = 19.33658$$

The solution of this model has a steady-state constraint,

$$a_2 = 5.0555 a_1$$

and the least square error is 3.090575. Values of γ_1 and γ_2 at convergence are found to be -6.676804×10^{-7} and 1.0000 respectively.

Comparing these two models of the same order of two different kinds of data, one can see from Figure 5 that the model of sinusoidal data has a faster rise time compared to the model of random data. The settling time for these two models is tabulated in Table 3. The settling time is defined as the time required for the transient to be within the specified percentage of the final value and remain within those limits.

Figure 7 is an optimum linear second-order solution for the random data. The differential equation identified is

$$\ddot{x} + a_1 \dot{x} + a_2 x = a_3 \quad (3.5)$$

TABLE III

SETTLING TIME

Order of the Model	2% of the steady state value		3% of the steady state value		5% of the steady state value	
	Random Data	Sinusoidal Data	Random Data	Sinusoidal Data	Random Data	Sinusoidal Data
1	1.3	1.02	1.17	0.92	1.0	0.8
2	0.69	0.66	0.676	0.607	0.64	0.5135
3	1.345	0.85	1.314	0.665	1.24	0.6
4	1.295	0.925	1.275	0.875	1.225	0.615

where

$$\begin{aligned} a_1 &= 6.760160 & \omega_n &= 4.66 \\ a_2 &= 21.77028 & \xi &= 6.726 \\ a_3 &= 110.0596 \end{aligned}$$

This linear model has a steady state constraint

$$a_3 = 5.0555 a_2$$

and the least square error is 32.6736. The second order model has five first order differential equations for the purpose of identification. The initial conditions $x(0)$ and $\dot{x}(0)$ are considered to be known and equal to zero. The parameter a_3 is linearly dependent on the parameter a_2 . Therefore, only two parameters, a_1 and a_2 are to be identified for this model and only three independent solutions are required.

The initial condition matrix as described by equation (B.12) is

$$\Phi(t_0) = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 \rho_3 & \alpha_3 & \alpha_3 \\ \alpha_4 & \alpha_4 \rho_4 & \alpha_4 \\ c\alpha_4 & c\alpha_4 \rho_4 & c\alpha_4 \end{bmatrix} \quad (3.6)$$

where

$$c = 5.0555$$

$$\rho_3 = \rho_4 = 1.2$$

γ_1 , γ_2 , and γ_3 at convergence are found to be -1.642473×10^{-4} , $-1.9062885 \times 10^{-4}$, and 1.000055 respectively.

In Figure 11 is shown an optimum solution for the sinusoidal data where

$$\ddot{x} + a_1 \dot{x} + a_2 x = a_3 \quad (3.7)$$

and

$$\begin{aligned} a_1 &= 17.89108 & \omega_n &= 8.125 \\ a_2 &= 79.49779 & \xi &= 1.089 \\ a_3 &= 401.9006 \end{aligned}$$

The steady state constraint is

$$a_3 = 5.0555 a_2$$

and the least square error is 1.21893. γ_1 , γ_2 , and γ_3 at convergence are found to be $-6.4505637 \times 10^{-3}$, $-8.4769838 \times 10^{-3}$ and 1.014927 respectively.

As shown for the first order models of both sets of data, the second order models also show that the rise time of the model of sinusoidal data is faster than the rise time of the model of random data. As shown in Table 3, the settling time is larger for identified models from random data than the settling time of identified models from the sinusoidal data. The natural frequency and damping ratio of the sinusoidal model is higher compared to the natural frequency and damping ratio of random model. The sinusoidal model is overdamped whereas the random model is underdamped with a 3.65% overshoot. The parameter a_2 is the square of natural frequency and the parameter a_1 is twice the product of natural frequency and damping ratio.

Figure 8 is an optimal least square solution of third order model for the random data where

$$\dot{x} + a_1 x = a_2, \quad \ddot{y} + b_1 \dot{y} + b_2 y = x \quad (3.8)$$

and

$$a_1 = 3.856122$$

$$a_2 = 960.3408$$

$$b_1 = 4.000962$$

$$b_2 = 49.26184$$

$$\omega_n = 7.018$$

$$\xi = 0.285$$

This third order model has a steady state constraint

$$a_2 = 5.0555 a_1 b_2$$

and the least square error is 30.81062. The third order model has seven first order differential equations for the purpose of parameter identification. The initial conditions $y(0)$, $\dot{y}(0)$, and $x(0)$ are considered to be known and equal to zero and the parameter a_2 is dependent on the parameters a_1 and b_2 . Therefore, only three parameters a_1 , b_1 , and b_2 are to be identified which requires only four independent solutions.

The initial condition matrix as described by equation (B.12) is

$$\Phi(t_0) = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 & \alpha_3 \\ \alpha_4 \rho_4 & \alpha_4 & \alpha_4 & \alpha_4 \\ \alpha_5 & \alpha_5 \rho_5 & \alpha_5 & \alpha_5 \\ \alpha_6 & \alpha_6 & \alpha_6 \rho_6 & \alpha_6 \\ c\alpha_5\alpha_6 & c\alpha_5\alpha_6\rho_5 & c\alpha_5\alpha_6\rho_6 & c\alpha_5\alpha_5 \end{bmatrix} \quad (3.9)$$

where

$$c = 5.0555$$

$$\rho_4 = \rho_5 = \rho_6 = 1.2$$

γ_1 , γ_2 , γ_3 , and γ_4 at convergence are found to be $-1.0435563 \times 10^{-4}$, $-1.4409055 \times 10^{-4}$, $-1.6509967 \times 10^{-4}$, and 1.000412 respectively.

Figure 12 is the optimum third order solution for the sinusoidal data where

$$\dot{x} + a_1 x = a_2, \quad \ddot{y} + b_1 \dot{y} + b_2 y = x$$

and

$$a_1 = 4.975478 \quad \omega_n = 17.7$$

$$a_2 = 7884.874 \quad \xi = 0.2942$$

$$b_1 = 10.2924$$

$$b_2 = 313.4702$$

The steady state constraint is

$$a_2 = 5.0555 a_1 b_2$$

and the least square error is 0.4333401. γ_1 , γ_2 , γ_3 , and γ_4 are

found to be 4.697885×10^{-5} , 7.291275×10^{-5} , 2.403890×10^{-4} , and 0.9995397 respectively.

Comparing the third order linear models, we notice that both of them have approximately the same damping ratio but the natural frequency of the sinusoidal model is higher than the natural frequency of the random model. Values of all the parameters of the sinusoidal model are larger than that of the random model.

Figure 9 is an optimum solution of fourth-order linear model for the random data where

$$\ddot{x} + a_1 \dot{x} + a_2 x = a_3, \quad \ddot{y} + b_1 \dot{y} + b_2 y = x \quad (3.10)$$

and

$a_1 = 25.78181$	$\omega_{n1} = 9.5$
$a_2 = 90.26985$	$\xi_1 = 1.355$
$a_3 = 25705.09$	$\omega_{n2} = 7.5$
$b_1 = 3.628691$	$\xi_2 = 0.242$
$b_2 = 56.32648$	

The least square error is 30.7365 for a steady state constraint of

$$a_3 = 5.0555 a_2 b_2$$

The fourth order model has nine first-order differential equations for parameter identification. The initial conditions $x(0)$, $\dot{x}(0)$, $y(0)$, and $\dot{y}(0)$ are considered to be known and equal to zero. The parameter a_3 is dependent on the parameters a_2 and b_2 . Since only four independent parameters are to be identified for this model, only five independent solutions are needed.

The new initial condition matrix is described by equation (B.12) as

$$\Phi(t_0) = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 & \alpha_3 & \alpha_3 \\ \alpha_4 & \alpha_4 & \alpha_4 & \alpha_4 & \alpha_4 \\ \alpha_5 \rho_5 & \alpha_5 & \alpha_5 & \alpha_5 & \alpha_5 \\ \alpha_6 & \alpha_6 \rho_6 & \alpha_6 & \alpha_6 & \alpha_6 \\ \alpha_7 & \alpha_7 & \alpha_7 \rho_7 & \alpha_7 & \alpha_7 \\ \alpha_8 & \alpha_8 & \alpha_8 & \alpha_8 \rho_8 & \alpha_8 \\ c\alpha_6\alpha_8 & c\alpha_6\alpha_8\rho_6 & c\alpha_8\alpha_6 & c\alpha_8\alpha_6\rho_8 & c\alpha_6\alpha_8 \end{bmatrix} \quad (3.11)$$

where

$$c = 5.0555$$

$$\rho_5 = \rho_6 = \rho_7 = \rho_8 = 1.2$$

$\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and γ_5 at convergence are found to be -1.770833×10^{-4} , $-7.3916581 \times 10^{-4}$, 5.156633×10^{-4} , 1.84303×10^{-3} , and 0.9985575 respectively.

Figure 13 is an optimum forth-order solution for the sinusoidal data. The differential equation is shown in equation (3.10) where

$$\begin{aligned} a_1 &= 7.274683 & \omega_{n_2} &= 17.22 \\ a_2 &= 296.2436 & \xi_2 &= 0.215 \\ a_3 &= 925558.2 & \omega_{n_1} &= 24.87 \\ b_1 &= 139.1458 & \xi_1 &= 2.8 \\ b_2 &= 618.0042 & & \end{aligned}$$

The least square error is 0.565178 and the steady state constraint is

$$a_3 = 5.0555 a_2 b_2$$

$\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and γ_5 are found to be -0.3066210, -0.3952708, 11.91018, 11.34555, and -21.55382. Values of $\gamma_1, \gamma_2, \gamma_3$, and γ_4 are not close to zero and γ_5 is not close to one. This search was terminated at this point since it was observed that further search for an optimum value did not improve the least square error.

In general, one can notice that the rise in the transient region is faster for all models of sinusoidal data compared to the rise of the corresponding model of random data. Numerical values of all the parameters of the models of sinusoidal data are higher when compared to the numerical values of the parameters of the models of random data, Table 4. In Table 5, the convergence constants γ_i are summarized for the different orders of models.

III. STEADY STATE CONSTRAINT

The linear models considered here with a step input will always reach a steady state value as time goes to infinity. This could be illustrated by taking a first order model.

$$\dot{x} + a_1 x = a_2 \quad (3.12)$$

The total solution of equation (3.12) is

$$x(t) = \frac{a_2}{a_1} (1 - e^{-a_1 t}) \quad (3.13)$$

TABLE IV

NUMERICAL VALUES OF IDENTIFIED PARAMETERS

(a) Random Data					
Order of Model	a_1	a_2	a_3	b_1	b_2
1	3.00	15.16			
2	6.76	21.77	110.06		
3	3.86	960.34		4.00	49.26
4	25.78	90.27	25705.09	3.63	56.33
(b) Sinusoidal Data					
Order of Model	a_1	a_2	a_3	b_1	b_2
1	3.82	19.33			
2	17.89	79.50	401.90		
3	4.98	7884.88		10.29	313.47
4	7.27	296.24	925558.2	139.14	618.00

TABLE V

 γ s FOR MODELS OF RANDOM DATA

Order of the Model	γ_1	γ_2	γ_3	γ_4	γ_5
1	-7.446891×10^{-5}	1.000074			
2	-1.642473×10^{-4}	$-1.9062885 \times 10^{-4}$	1.000055		
3	$-1.0435563 \times 10^{-4}$	$-1.4409055 \times 10^{-4}$	$-1.6509967 \times 10^{-4}$	1.000412	
4	-1.770833×10^{-4}	$-7.3916581 \times 10^{-4}$	5.156633×10^{-4}	1.843031×10^{-3}	0.9985575

 γ s FOR MODELS OF SINUSOIDAL DATA

Order of the Model	γ_1	γ_2	γ_3	γ_4	γ_5
1	$-6.6768041 \times 10^{-7}$	1.000000			
2	$-6.4505637 \times 10^{-3}$	$-8.4769838 \times 10^{-3}$	1.014927		
3	4.697885×10^{-5}	7.291275×10^{-5}	2.4038908×10^{-4}	0.9996397	
4	-0.3066210	-0.3952708	11.91018	11.34555	-21.55382

where

$$x(0) = 0.0$$

Assuming a_1 to be a positive number, for $t \gg 0$

$$x(t) \rightarrow \frac{a_2}{a_1} = c = \text{constant} \quad (3.14)$$

or

$$a_2 = c a_1$$

where c is a constant which is the steady state value.

All of the models that are considered have a steady-state value equal to 5.0555. This particular value of steady state was chosen from the random data shown in Figure 5. The random data has random oscillations about 5.0555 after the first one-second time period. The convergence space of this method of parameter identification is defined to be those values of parameters which will converge to the optimum parameters. The convergence space is increased by use of the steady state constraint. In the case of the random data, the steady state constraint will force the solution to the steady state value and prevent large oscillations in the transient region which would have occurred in an attempt to match the random oscillations of the data.

The steady state constraint will cause a large error for the models of random data in the steady state region, but this error is justified by a decreased least square error in the transient region.

IV. LEAST SQUARE ANALYSIS

The method of least square (Appendix C) provides an indication of the degree of accuracy of the fitted curve to the original data. This degree of accuracy or approximation is dependent on the least square error. Figure 14 shows the least square error for all the different models. Figure 14a is for random data which shows that the least square error is exponentially decreasing as the order of a model is increased up to a certain order. The difference in the least square error of third order model and fourth order model is not significant. For both sets of data, the same results are achieved. The least square error may decrease as the order of the model is increased, but this will also increase the dimension of the matrix that has to be inverted, and matrix inversion on the digital computer is inefficient compared to other numerical operations. This is a source of an error which may cause the least square error to increase instead of decreasing.

From the results presented in Table 6, we can conclude that the third order models are the optimum models for both sets of data.

The least square error for all of the random models is much higher than the models of sinusoidal data. This difference in least square errors is in the steady state region. The least square error in the steady-state region for the models of sinusoidal data is negligible whereas least square error in the steady-state region of the models of random data is very high.

The error encountered in linear models is presented in Figure 15 through 22. This error could be divided into two parts: (1) error

TABLE VI

LEAST SQUARE ERROR

Order of a System	Sinusoidal Data (34 B.C.)	Random Data (35 B.C.)
1	3.090595	38.13594
2	1.21893	32.6736
3	0.4333401	30.81065
4	0.56517	30.7364

in the transient region and (2) error in the steady-state region. As discussed before, error in the steady-state region remains constant. Error in the steady-state region is caused by the steady-state constraint which forces the steady-state value of an identified model to a constant value. Error in the transient region is decreasing as the order of a model is increased. This is observed by noting the change in the envelope of the error in the transient region. For higher order models, this envelope becomes smaller which is the indication of a better fit to the data.

V. CONVERGENCE OF THE IDENTIFICATION SCHEME

Kalaba (1965) has proven that if the iterative process of quasilinearization converges to a solution it does so quadratically in the neighborhood of the solution. Practically, this means that when the parameter converges close to its optimum value, the number of significant digits in the approximate solution is at least doubled with each iteration. This property of convergence of a first-order model to the sinusoidal data is considered, the differential equation of the first order model is

$$\dot{x} + a_1 x = a_2 \quad (3.15)$$

where a_2 is constrained by the relationship

$$a_2 = 5.0555 a_1$$

The initial guess for a_1 is 5.368028. Table 7 shows that each iteration is improving the parameter a_1 and that the change in the parameter a_1 is approaching to zero. It may be observed that the

TABLE VII

CONVERGENCE OF PARAMETERS ¹

No. of Iteration	Change in Parameter a_1	Parameter a_1
		5.368028 ²
1	-2.277236	3.090792
2	0.69425200	3.785044
3	0.004498195	3.830026
4	-0.000595474	3.824071
5	0.000846862	3.824918
6	-0.000012683	3.824719
7	0.000022888	3.824814
8	0.000006675	3.824821
9	-0.000003337	3.824817

¹
Identification of the first order model

$$\dot{x} + a_1 x = a_2$$

where $a_2 = 5.0555 a_1$

²
Initial guess for a_1

number of correct significant digits of the parameter is not exactly doubled each time. This fact is the result of round-off errors encountered in calculations. The program used for the identification was used in single precision arithmetic and the accuracy after six or seven significant digits is not guaranteed. Double precision arithmetic should provide better results.

Divergence due to improper initial guess is shown in Figure 2. A second order model with a differential equation

$$\ddot{x} + a_1 \dot{x} + a_2 x = a_3 \quad (3.16)$$

is considered. The initial values of a_1 , a_2 , a_3 are 8.5, 44.0, 100.0 respectively. Figure 2 shows that the search for an optimum point is not in the right direction. Every iteration increases the value of parameters. The same second order model with a different initial guess and a constraint $a_3 = 5.0555 a_2$ is shown in Figure 3, where the search for an optimum point is shown. It is easy to see how the solution proceeds towards an optimum point.

The path of convergence can be changed as follows:

$$\eta_i = \beta \alpha_i$$

where $0 < |\beta| < 1$. Luckinbill and Childs (1968). This factor β will alter the change in initial condition vector from α_i to η_i . The value of β could be automatically adjusted to meet the required change in the initial condition vector. This is illustrated in Figure 4 where a fourth order model is considered. The differential equations are

$$\ddot{x} + a_1 \dot{x} + a_2 x = a_3, \quad \ddot{y} + b_1 \dot{y} + b_2 y = x$$

The initial guesses for a_1 , a_2 , b_1 , and b_2 are 6.082, 304.708, 69,351 and 295.603 respectively, where

$$a_3 = 5.0555 \, b_2 a_2$$

In Figure 4, b_1 vs. b_2 is plotted, where the dotted line shows the proposed change. This is the search trajectory of the parameters b_1 vs. b_2 without scaling by the factor. The solid line in Figure 4 is the search trajectory using a factor β . It is seen that the factor β significantly alters the trajectory of the parameters.

CHAPTER IV

I. CONCLUSIONS

Identification of parameters in a linear system by the method of quasilinearization has been proven feasible in a number of previous papers and books. However, this study has endeavored to show that an optimum order of a linear model does exist for a given numerical technique, data set and digital computer with which the identification is performed.

For the IBM-360/40, single precision arithmetic, Runge-Kutta fourth order integration of step size .01 and Gauss Siedel matrix inversion, it has been shown that the third order model is an optimum model for the given random and sinusoidal data. As stated by Luckinbill and Childs (1968), an explicit indication of the numerical round-off error which is occurring in the identification procedure is given by γ_{i+1} for the $(i+1)$ model as shown in Table 4. As the order of the model to be identified increases, the number of parameters and unknown initial conditions increase which causes the dimension of the matrix to be inverted to increase and numerical round-off error in integration to increase. As the number of boundary conditions increase above the number required, the numerical round off error will also increase due to the matrix multiplications which are involved in the least square fitting of the data to the model.

To maximize the usefulness of the quasilinearization method of identification, it would be best to identify all of the parameters in the model with no constraints on the parameters. However, it has been

found in this study that the steady state constraint alteration of new initial conditions and a non-negative restraint on the parameters or initial conditions were necessary in order to ensure convergence to an optimum least square solution.

The use of the steady state constraint requires that the identification be an off-line procedure due to the need of knowing the expected or steady state value of the data after the transients have died out.

II. EXTENSIONS AND FURTHER WORK

The main problem encountered in the numerical experiments performed was finding an initial guess which would converge to the optimum solution. Since it is not difficult to find an initial guess for the first order models which will converge without the need for constraints, it is recommended that a procedure be found for increasing the order of the linear system one parameter at a time until the least square error stops decreasing. For example, assume that we have identified a set of parameters, a_1 and a_2 in the first order model.

$$\dot{x} + a_1 x = a_2$$

Next we are to select an initial guess for the parameters b_1 and b_2 in the second order model

$$\ddot{x} + b_1 \dot{x} + b_2 x = b_3$$

It may be possible that if b_1 and b_2 are selected such that one of the characteristic roots of the second-order equation is equal to $-a_1$ and the other root is chosen to be 10 times $-a_1$ then the identification

process will converge rapidly to the solution. No such approach has been investigated here but such an approach may prove valuable. For instance, it may be possible to identify the steady-state value of the data by using the ratio a_2/a_1 from an unconstrained 1st order model and to choose and constrain the 2nd and all higher order models.

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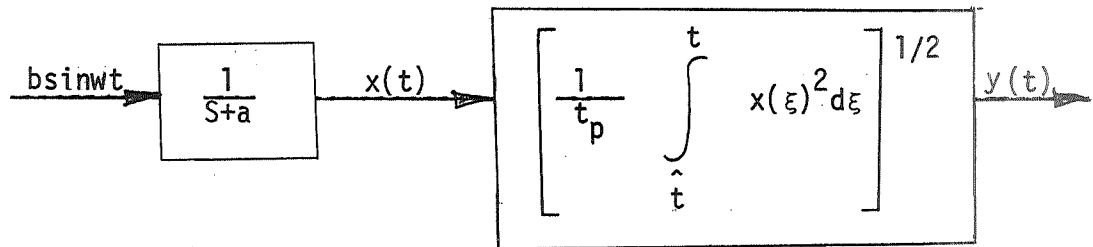
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APPENDICES

APPENDIX A

FIRST ORDER SYSTEM AND RMS DETECTOR

The following system is to be solved for an analytical solution

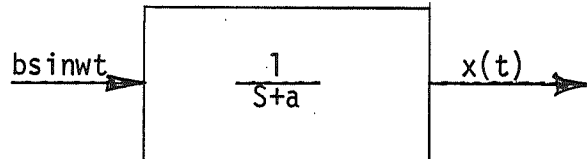


where a and b are constants

$$t_p = t - (t-T)U(t-T)$$

$$\hat{t} = (t-T)U(t-T)$$

Consider the following system



This system could be expressed as

$$\dot{x} + ax = b \sin \omega t \quad (A.1)$$

The solution of equation (A.1) is

$$x(t) = \frac{1}{\sqrt{a^2 + \omega^2}} \left[\frac{\omega}{\sqrt{a^2 + \omega^2}} e^{-at} - \cos(\omega t + \phi) \right] \quad (A.2)$$

where

$$\phi = \tan^{-1}\left(\frac{a}{\omega}\right)$$

Let $z = wt + \phi$ so that

$$\begin{aligned} \frac{1}{t_p} \int_{\hat{t}}^t x^2(\xi) d\xi &= \frac{b^2}{t_p(a^2 + w^2)} \left[\frac{w^2}{-2a(w^2 + a^2)} e^{-2at} \right] \Big|_{\hat{t}}^t \\ &- \frac{2w}{\sqrt{w^2 + a^2}} \frac{e^{-a/w Z} e^{a \frac{\phi}{w}}}{\left(\frac{a^2}{w^2}\right) + 1} \left\{ -a/w \cos z + \sin z \right\} \Big|_{w\hat{t} + \phi}^{wt + \phi} \\ &+ \frac{1}{w} \left\{ \frac{Z}{2} + \frac{\sin 2Z}{4} \right\} \Big|_{w\hat{t} + \phi}^{wt + \phi} \end{aligned}$$

Simplifying the above equation

$$\begin{aligned} y(t) &= \left[\frac{b^2}{t_p(a^2 + w^2)} \right]^{1/2} \left[\frac{-w^2}{2a(w^2 + a^2)} \left\{ e^{-2at} - e^{-2a\hat{t}} \right\} \right. \\ &- \frac{2w^2}{(w^2 + a^2)^{3/2}} \left\{ e^{-at} \left(-\frac{a}{w} \cos(wt + \phi) + \sin(wt + \phi) \right) \right. \\ &- \left. \left. e^{-a\hat{t}} \left(-\frac{a}{w} \cos(w\hat{t} + \phi) + \sin(w\hat{t} + \phi) \right) \right\} \right. \\ &+ \left. \left\{ \frac{(t - \hat{t})}{2} + \frac{1}{4w} (\sin 2(wt + \phi) - \sin 2(w\hat{t} + \phi)) \right\} \right]^{1/2} \end{aligned}$$

(A.3)

Letting $\hat{t} = 0$

$$\begin{aligned}
 y(t) = & \left[\frac{b^2}{t(a^2 + w^2)} \right]^{1/2} \left[\frac{-w^2}{2a(w^2 + a^2)} (e^{-2at} - 1) \right. \\
 & - \frac{2w^2}{(w^2 + a^2)^{3/2}} \left\{ e^{-at} \left(-\frac{a}{w} \cos(wt + \phi) + \sin(wt + \phi) \right) \right. \\
 & \left. \left. - \left(-\frac{a}{w} \cos \phi + \sin \phi \right) \right\} + \frac{t}{2} + \frac{1}{4w} (\sin 2(wt + \phi) - \sin 2\phi) \right]^{1/2}
 \end{aligned}
 \tag{A.4}$$

It is easily shown that as $t \rightarrow \infty$

$$y(t) = \left[\frac{b^2}{2(a^2 + w^2)} \right]^{1/2}
 \tag{A.5}$$

APPENDIX B

SUPERPOSITION OF PARTICULAR SOLUTION

The work shown in this appendix is described in more detail by Luckinbill and Childs (1968).

Let us consider an nth order differential equation

$$\frac{d\bar{x}}{dt} = A\bar{x} + \bar{B} \quad (B.1)$$

where \bar{x} = vector with n elements

A = matrix with n x n elements which may be a function of x.

\bar{B} = vector with n elements

Let

$$x = \Phi_p \bar{\gamma} \quad (B.2)$$

where Φ_p is the solution matrix of equation (B.1) with a rank n and

$$\sum_{i=1}^{n+1} \gamma_i = 1 \quad (B.3)$$

Φ_p could be written as follows in the column vector form

$$\Phi_p = \left[\begin{array}{c|c|c|c} \bar{P}_1 & \bar{P}_2 & \text{-----} & \bar{P}_{n+1} \end{array} \right] \quad (B.4)$$

where \bar{P}_1 is a vector with n elements and

$$\bar{\gamma} = \left[\begin{array}{c|c|c|c} \gamma_1 & \gamma_2 & \text{-----} & \gamma_{n+1} \end{array} \right]^T \quad (B.5)$$

ϕ_p will satisfy the following differential equation

$$\frac{d\phi_p}{dt} = A \phi_p + S \quad (B.6)$$

where $S = (n \times n+1)$ matrix as shown below

$$S = \begin{bmatrix} \bar{B} & | & \bar{B} & | & \cdots & | & \bar{B} \end{bmatrix} \quad (B.7)$$

Substituting equation (B.2) into equation (B.1) and subtracting $S\bar{\gamma}$ from both sides,

$$\left[\frac{d\phi_p}{dt} - A\phi_p - S \right] \bar{\gamma} = \bar{B} - S\bar{\gamma} \quad (B.8)$$

Substituting equation (B.6) into equation (B.8) will give

$$S\bar{\gamma} = \bar{B} \quad (B.9)$$

This could be written as

$$B_j \sum_{i=1}^{n+1} \gamma_i = B_j \quad 1 \leq j \leq n$$

or

$$\sum_{i=1}^{n+1} \gamma_i = 1 \quad (B.10)$$

which is the same as equation (B.3).

By choosing proper initial conditions, \bar{p}_{n+1} will satisfy the boundary conditions, then

$$\bar{p}_{n+1} = \Phi_p' \bar{\gamma}' + \bar{p}_{n+1} \gamma_{n+1} \quad (B.11)$$

where

$$\Phi_p = \left[\begin{array}{c|c} \Phi_p' & \bar{p}_{n+1} \end{array} \right]$$

$$\bar{\gamma} = \left[\begin{array}{c|c} \bar{\gamma}' & \gamma_{n+1} \end{array} \right]$$

Substituting equation (B.3) into equation (B.11) will give

$$\bar{p}_{n+1} = \Phi_p' \bar{\gamma}' + \bar{p}_{n+1} - \bar{p}_{n+1} \sum_{i=1}^n \gamma_i$$

$$0 = \left[\begin{array}{cccc} \bar{p}_1 - \bar{p}_{n+1} & \bar{p}_2 - \bar{p}_{n+1} & \text{-----} & \bar{p}_n - \bar{p}_{n+1} \end{array} \right] \bar{\gamma}'$$

Since the matrix is not singular, $\bar{\gamma}'$ must be zero at convergence.

Let α be the vector which represents the initial conditions of the solution \bar{p}_{n+1} , and let ρ_i represent the perturbation of the i th initial condition. Then initial condition matrix will be

$$\Phi_p(t_0) = \left[\begin{array}{cccc|c} \alpha_1 \rho_1 & \alpha_1 & \text{----} & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \rho_2 & \text{----} & \alpha_2 & \alpha_2 \\ \vdots & \vdots & \text{----} & \vdots & \vdots \\ \alpha_n & \alpha_n & \text{----} & \alpha_n \rho_n & \alpha_n \end{array} \right] \quad (B.12)$$

By successively subtracting the last column from each of the other

columns, etc., one can reduce the matrix $\Phi_p(t_0)$ to a diagonal matrix with diagonal elements $\alpha_i \rho_i$. Therefore $\Phi_p(t)$ has a rank n .

APPENDIX C

LEAST SQUARE ESTIMATION

To illustrate the method of least squares, consider fitting a differential equation

$$\dot{y} + a_1 y = a_2 \quad (C.1)$$

to n given data points $x(t_1), x(t_2), \dots, x(t_n)$

The solution of equation (C.1), where $a_2 = ca_1$, is

$$y(t) = \frac{a_2}{a_1} (1 - e^{-a_1 t}) \quad (C.2)$$

where $y(t) = 0.0$ at $t = 0.0$

Since only one boundary condition is required to determine a_1 , $n > 1$ forms an overdetermined boundary value problem. In this case $y(t)$ will not pass through all the boundary values or it may not pass through any one of them. Therefore for a given value of a_1

$$y(t_i) = x(t_i) - \delta_i$$

where δ_i is the error.

Taking δ_i for each point and summing the squares of δ_i

$$L = \sum_{i=1}^n \delta_i^2 = \sum_{i=1}^n h_i \left\{ y(t_i) - x(t_i) \right\}^2 \quad (C.3)$$

Equation (C.3) could be written as

$$L = \sum_{i=1}^n h_i \left\{ \frac{a_2}{a_1} (1 - e^{-a_1 t_i}) - x(t_i) \right\}^2$$

where h_i is the weighting factor of each point. Since the absolute error is important in this problem let

$$h_1 = h_2 = h_3 = \dots = h_n = 1$$

Where relative accuracy is important, a choice of

$$h_i = 1/|x(t_i)|$$

could be made.

L is the measure of how well the curve fits the given points. If L is equal to zero, this means that all the given points lie on the curve of $y(t)$. If the points are away from $y(t)$, L will be larger. The problem is to find the parameter a_1 such that L will be minimum.

$$\frac{\partial L}{\partial a_1} = 0$$

The parameter a_2 is dependent on a_1 since $a_2 = ca_1$. Therefore an optimum value of a_1 will give an optimum value of a_2 .

$$\therefore \frac{\partial L}{\partial a_1} = 2 \sum_{i=1}^n \left[c(1 - e^{-a_1 t_i}) - x(t_i) \right] (c t_i e^{-a_1 t_i}) = 0$$

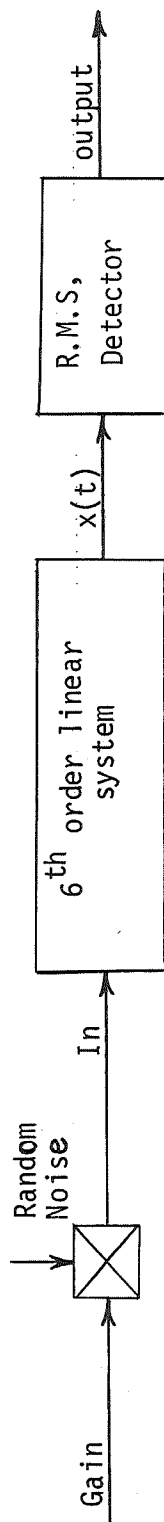
Dividing both sides of the above equation by $2c$

$$\frac{\partial L}{\partial a_1} = \sum_{i=1}^n \left[c(1 - e^{-a_1 t_i}) - x(t_i) \right] (t_i e^{-a_1 t_i}) = 0 \quad (C.4)$$

Equation (C.4) is a nonlinear equation for estimating one variable. When equation (C.1) is of higher order and more than one parameter is unknown equation (C.4) is replaced by a set of nonlinear simultaneous equations.

APPENDIX D

FIGURES



Transfer function of the 6th order system

$$\frac{x(t)}{In} = \frac{KS^3}{(s^2+b_1s+c_1)(s^2+b_2s+c_2)(s^2+b_3s+c_3)}$$

where

$$K = 6000 \times 10^7$$

$$b_1 = 20.28$$

$$c_1 = 1.962 \times 10^4$$

$$b_2 = 30.98$$

$$c_2 = 2.16 \times 10^4$$

$$b_3 = 35.22$$

$$c_3 = 1.989 \times 10^4$$

Figure 1. Physical System

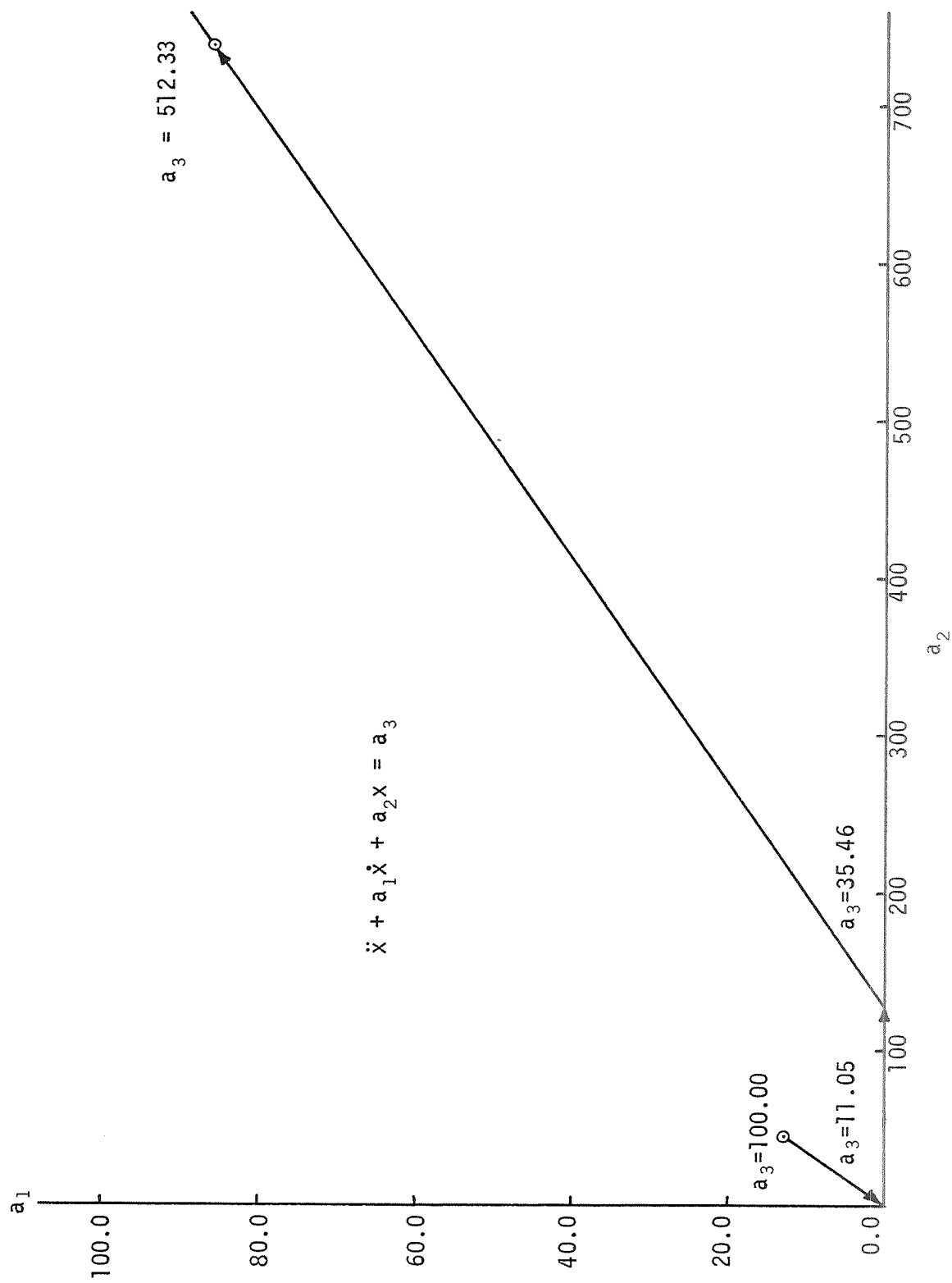


Figure 2. Example of Divergence of the Coefficients of a Second Order Model

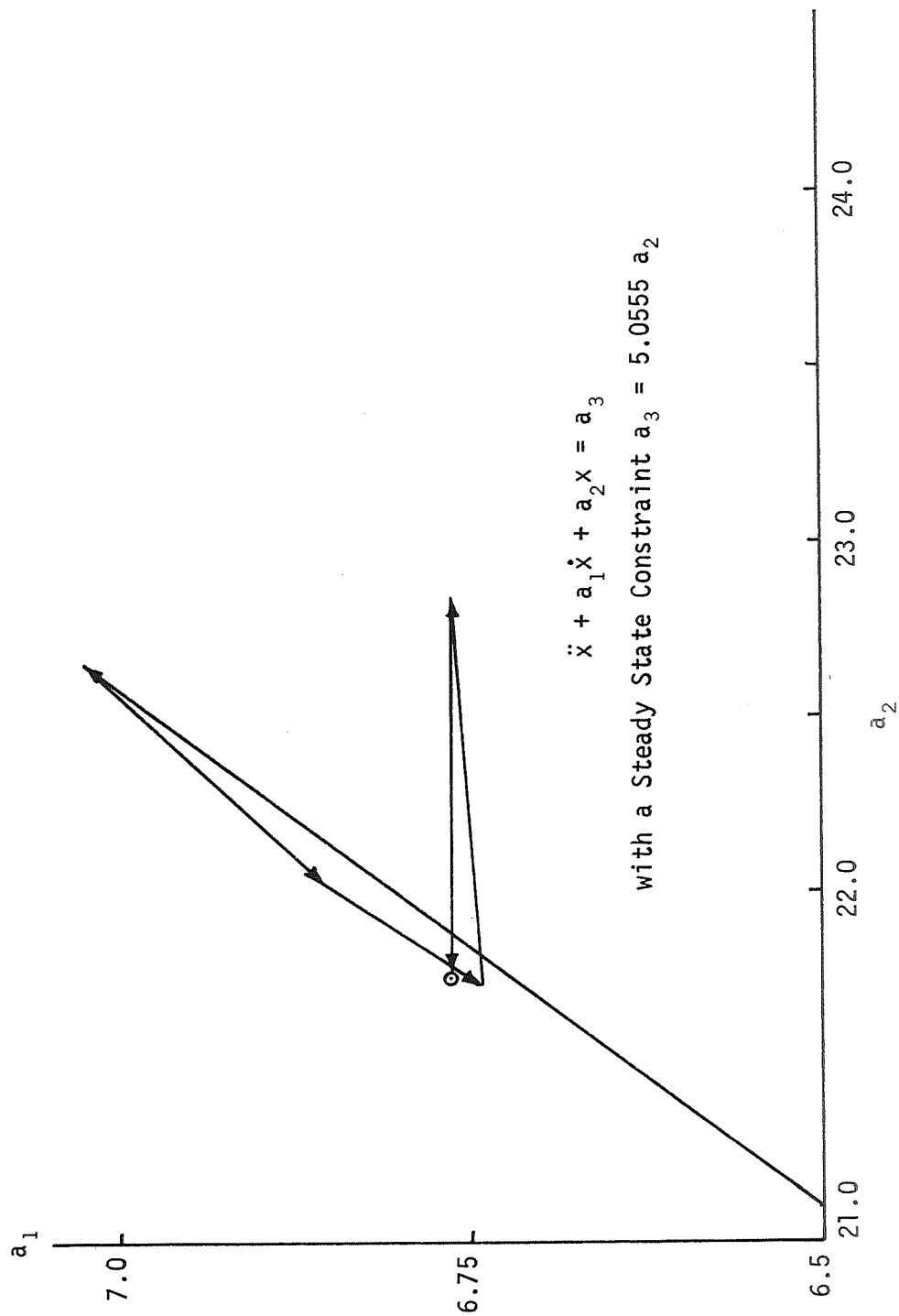


Figure 3. Example of Convergence of Coefficients of a Second Order Model

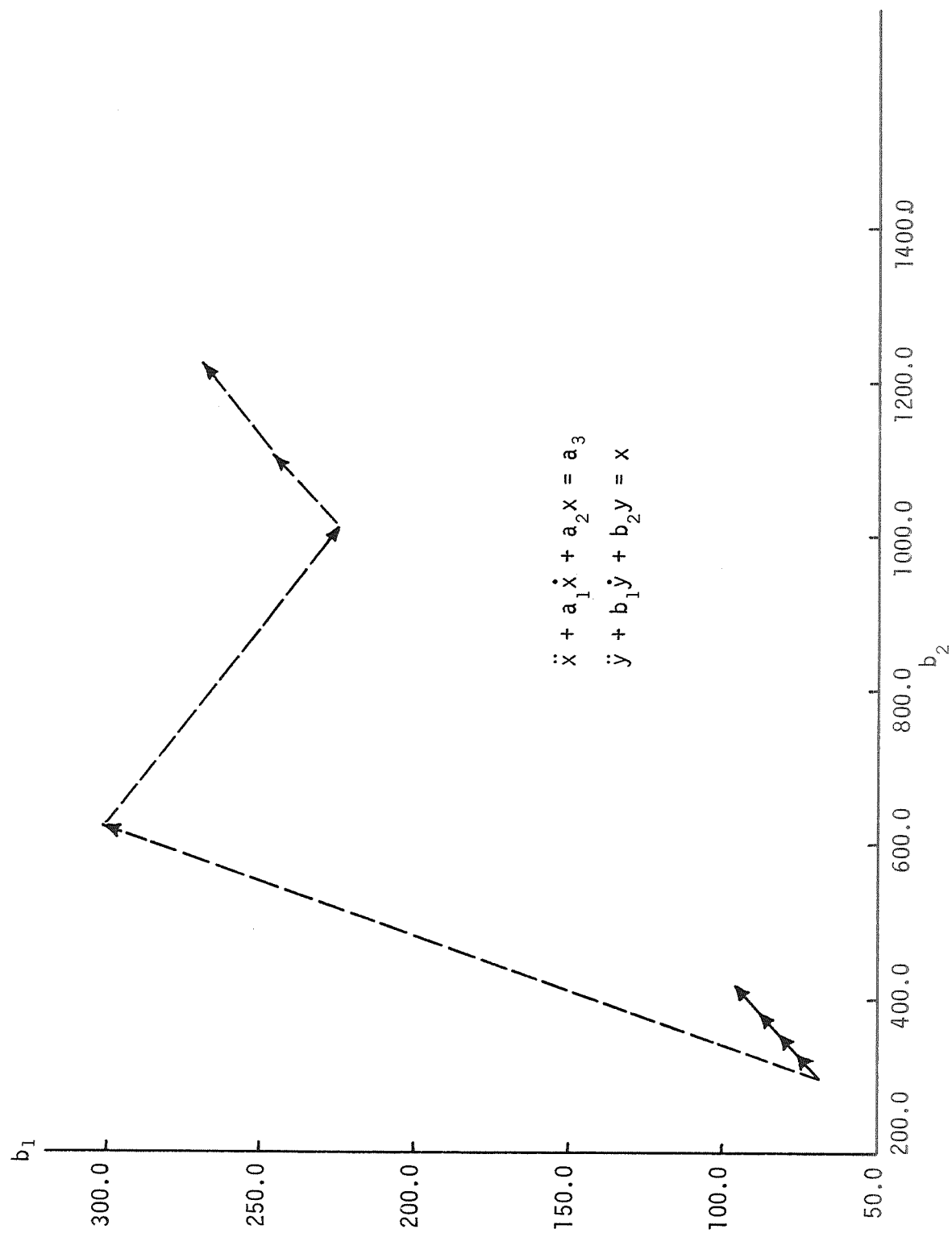


Figure 4. Altered Convergence of Coefficients of a Fourth Order Model

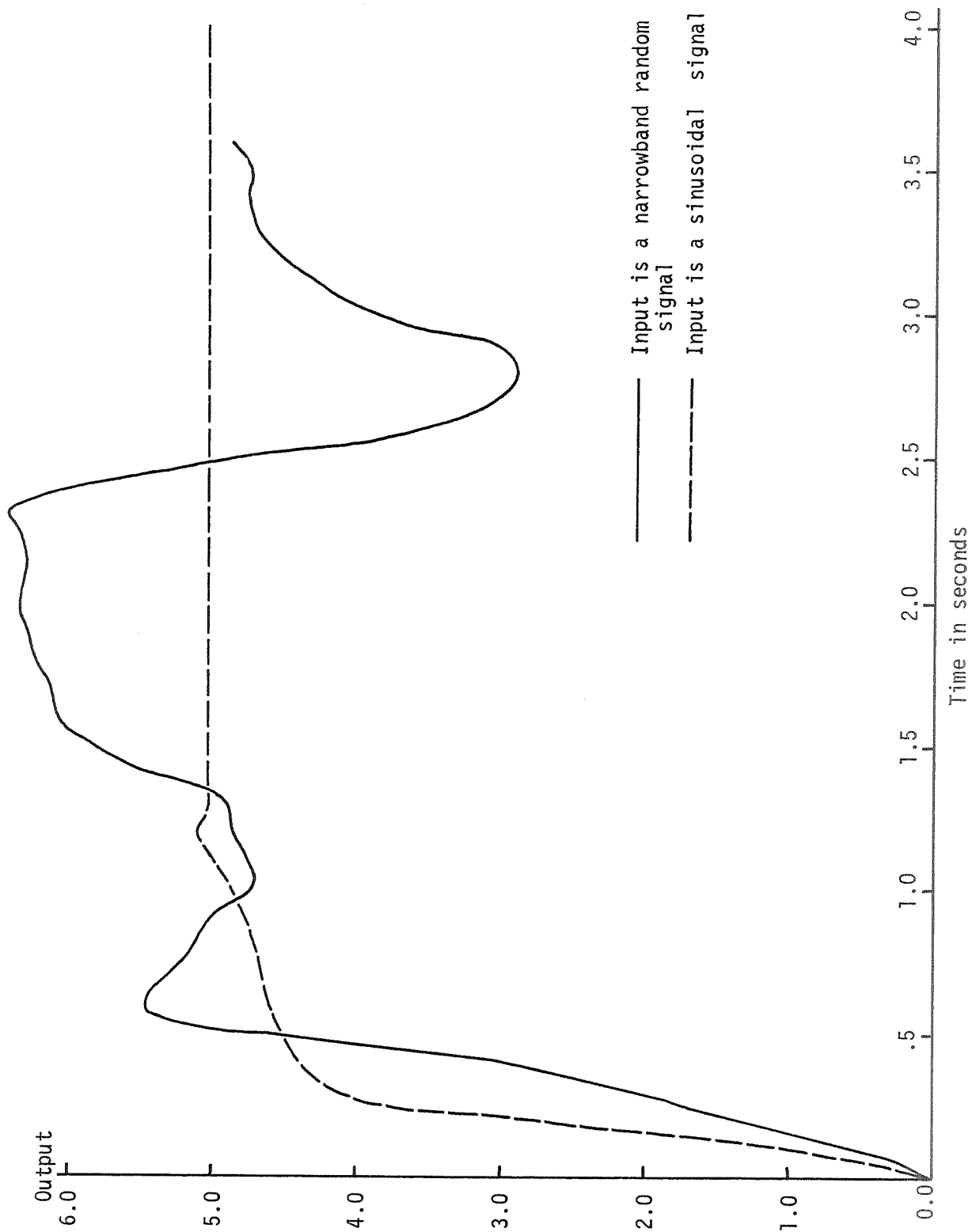


Figure 5. Output Data From the Physical System Shown in Figure 1 with Two Types of Input Signals

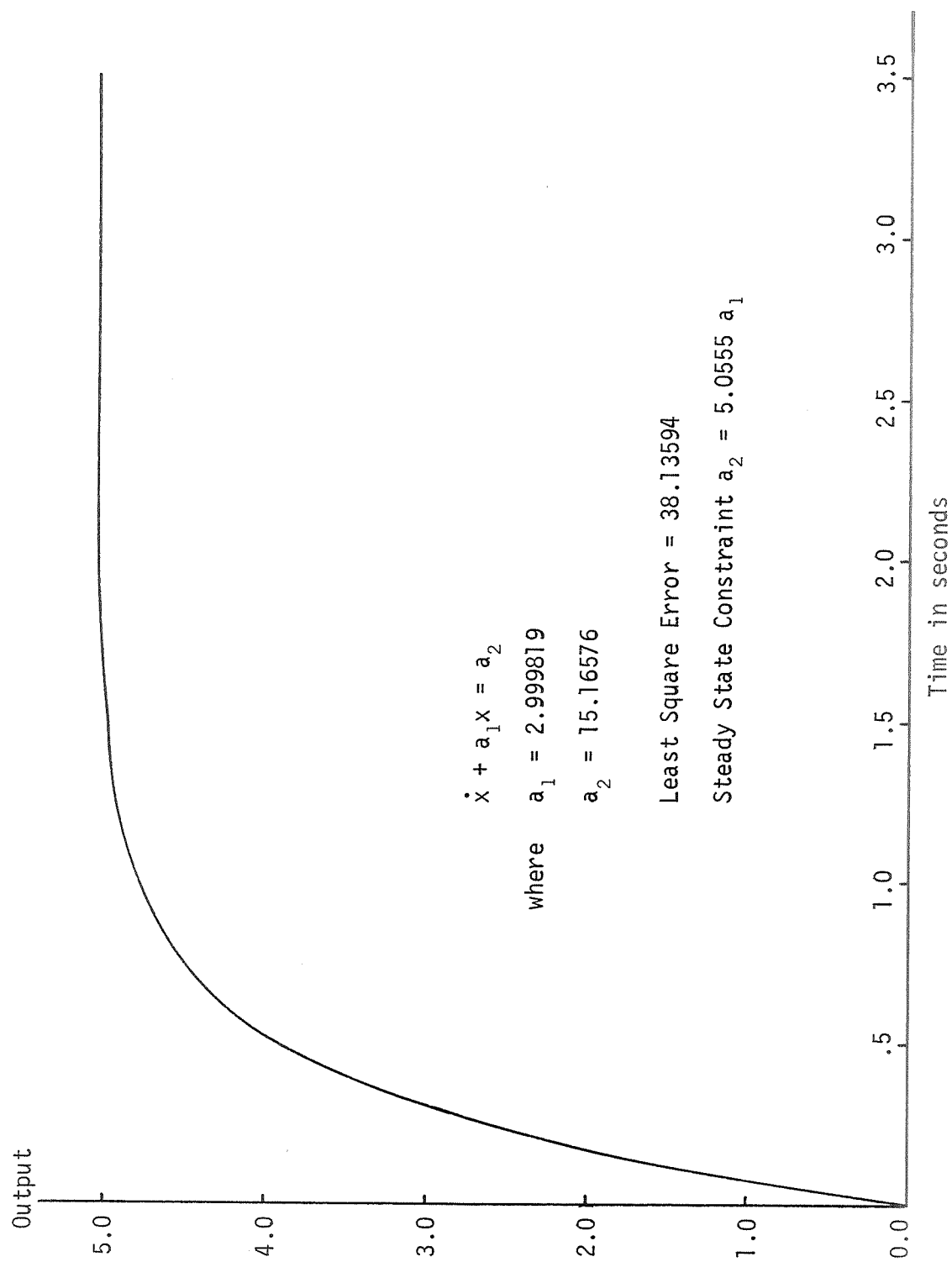


Figure 6. Linear First Order Model of Random Data Shown in Figure 5

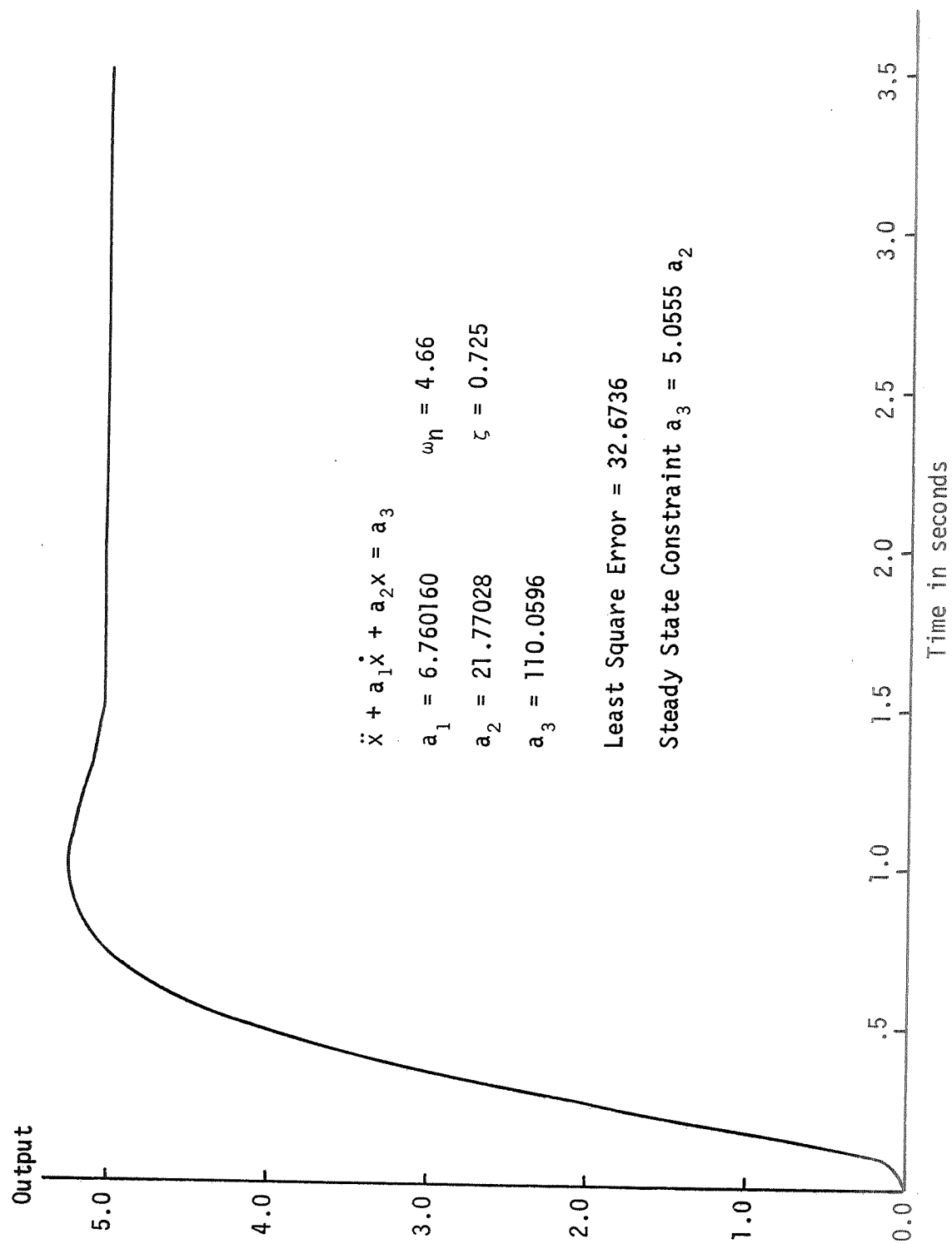


Figure 7. Linear Second Order Model of Random Data Shown in Figure 5

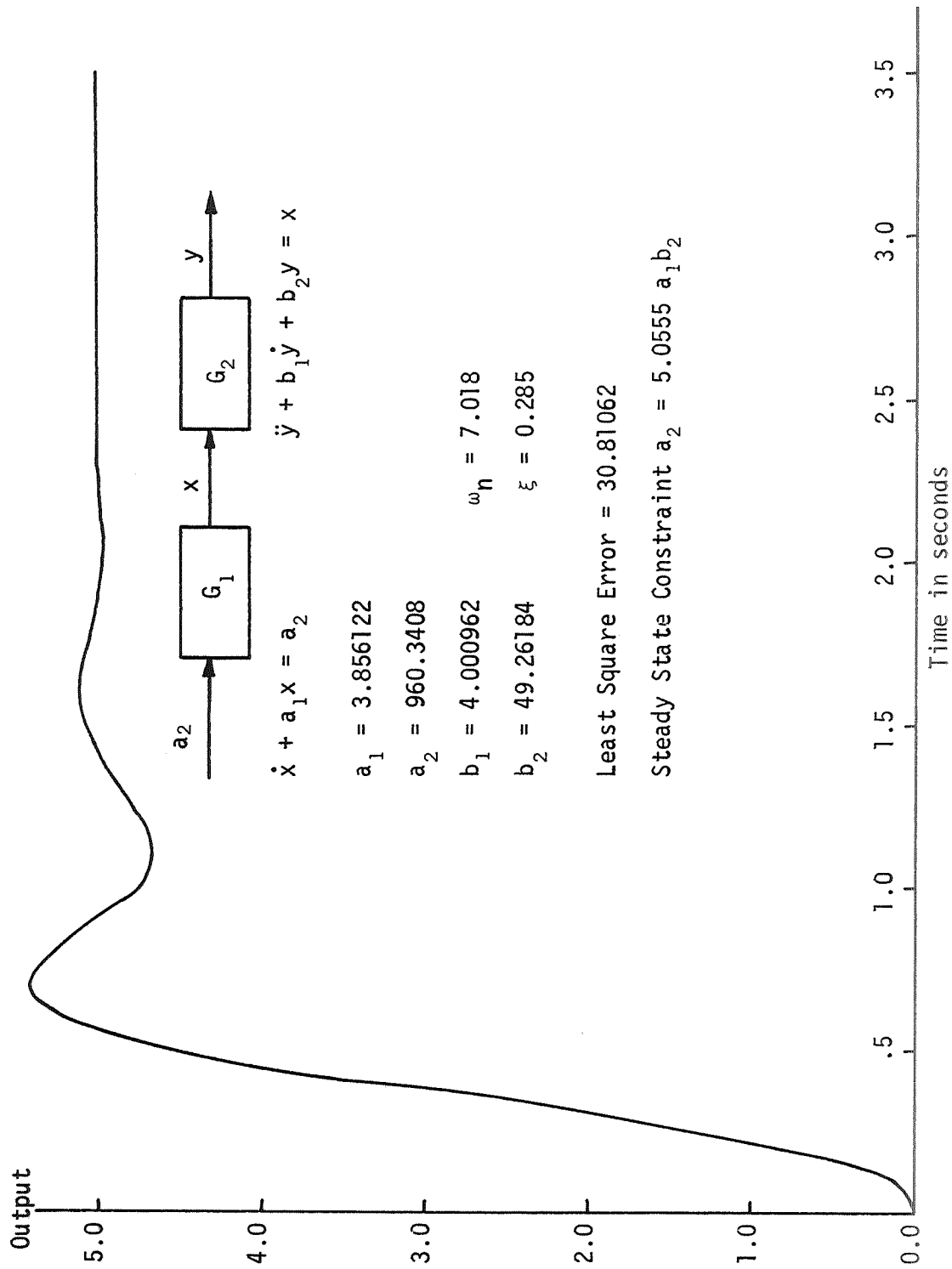


Figure 8. Linear Third Order Model of Random Data Shown in Figure 5

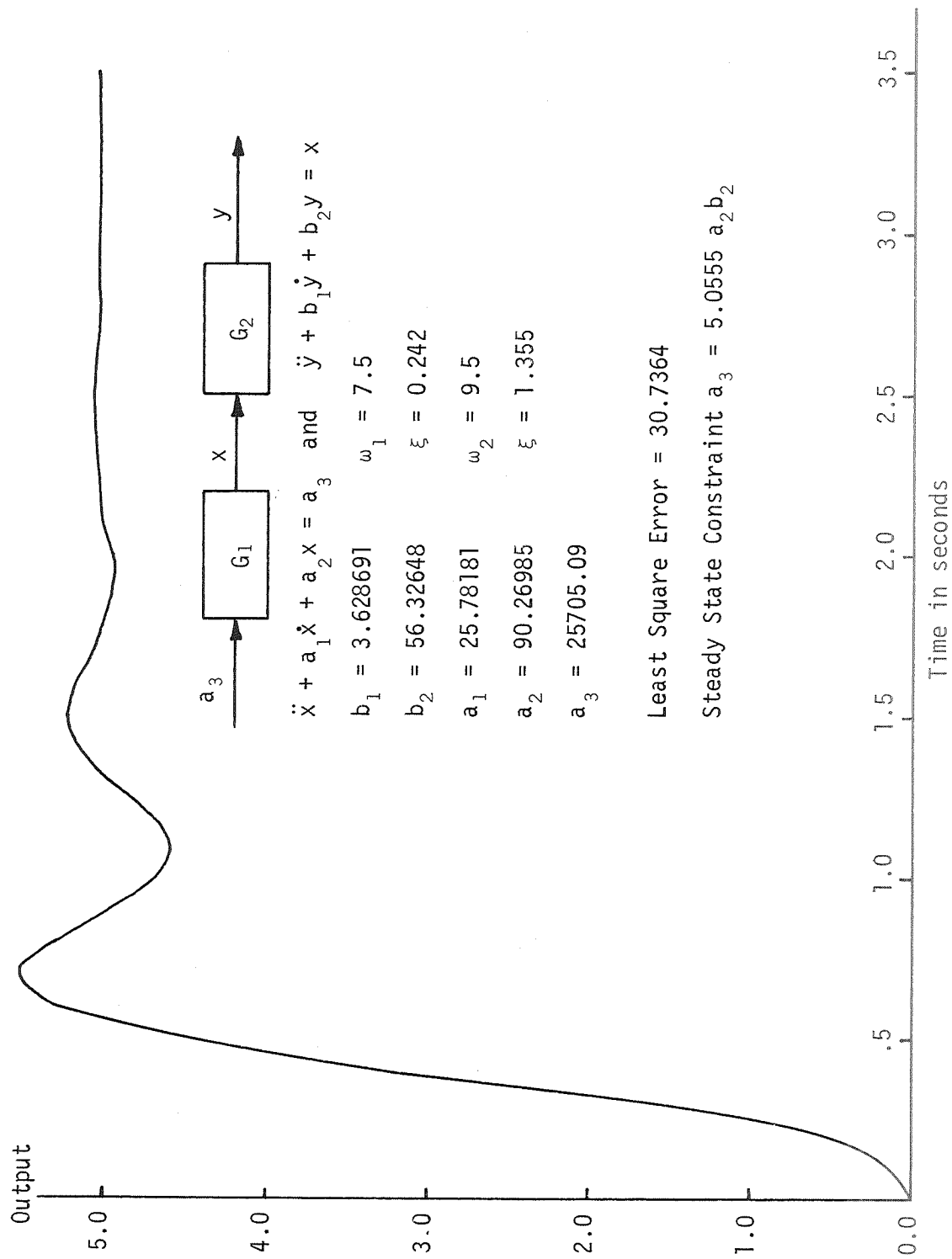


Figure 9. Linear Fourth Model of Random Data Shown in Figure5

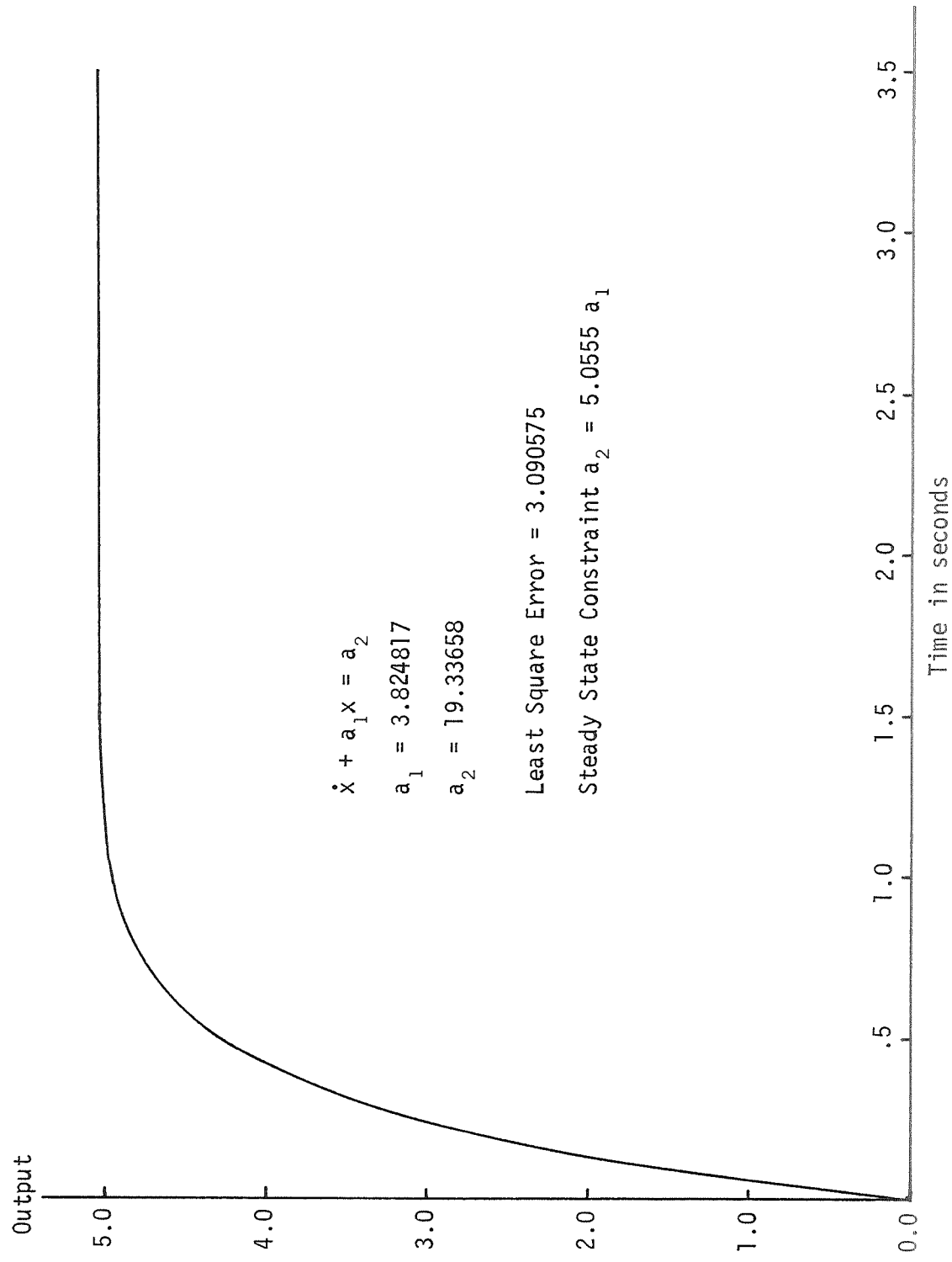


Figure 10. Linear First Order Model of Sinusoidal Data Shown in Figure 5

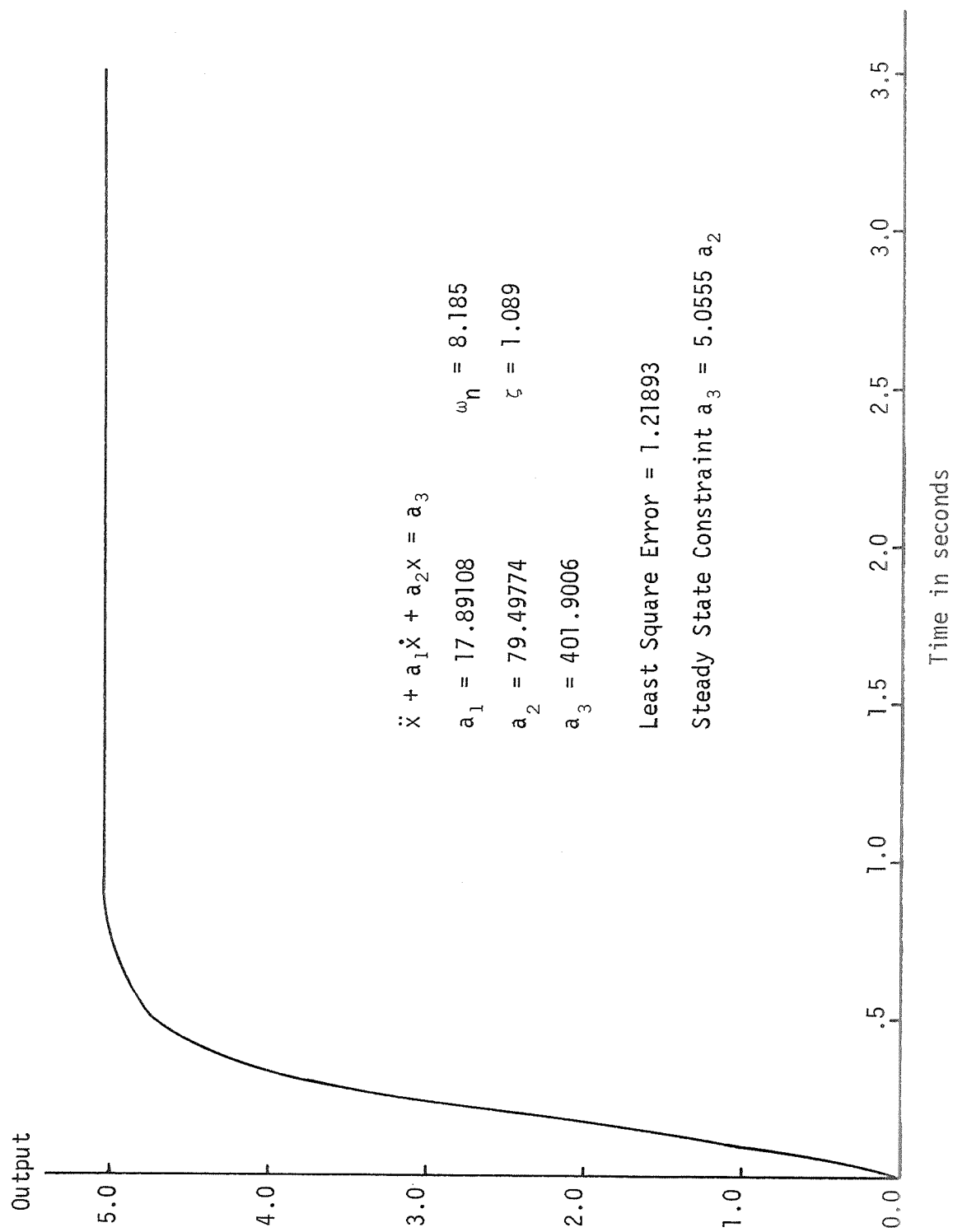


Figure 11. Second Order Model of Sinusoidal Data Shown in Figure 5

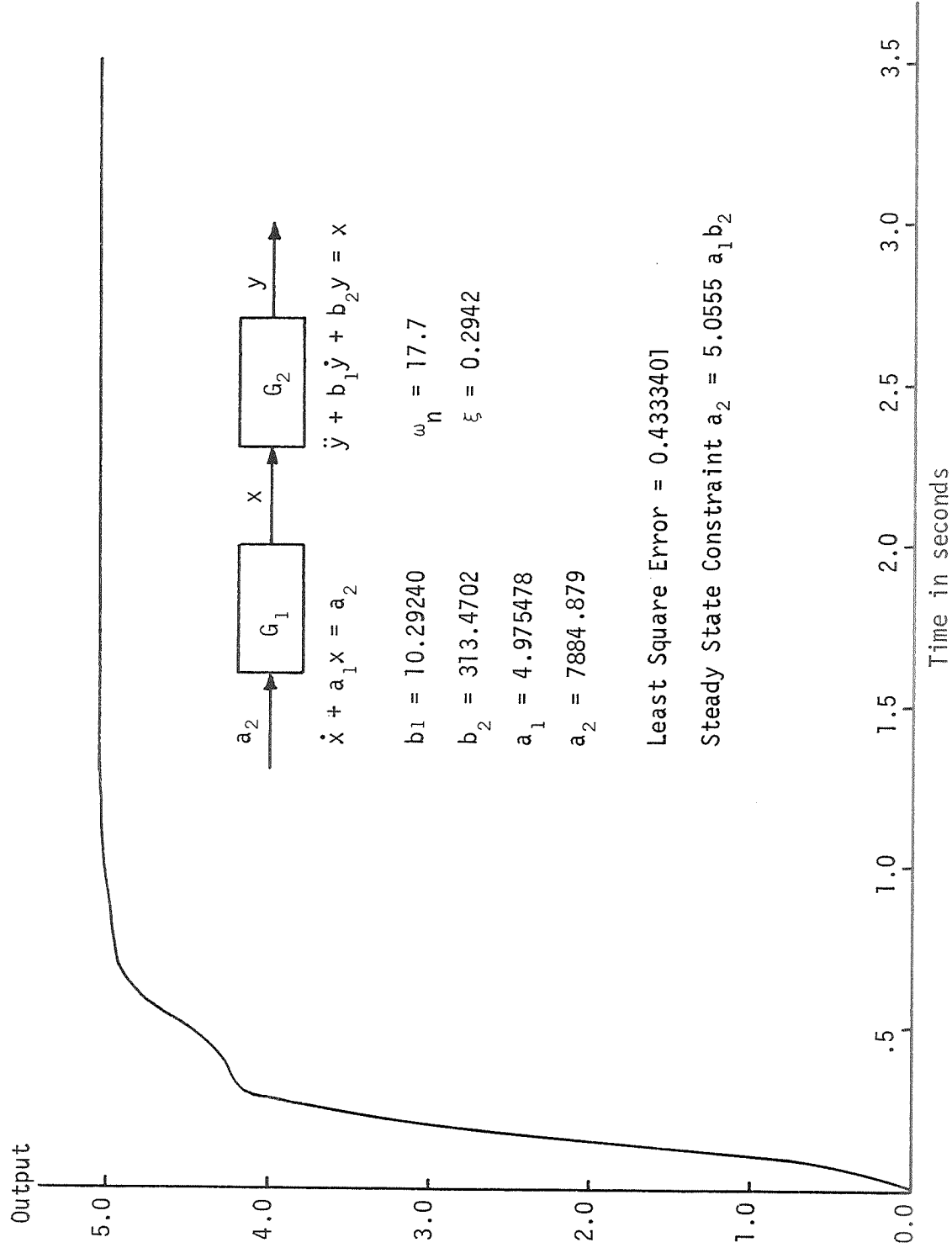


Figure 12. Third Order Model for Sinusoidal Input

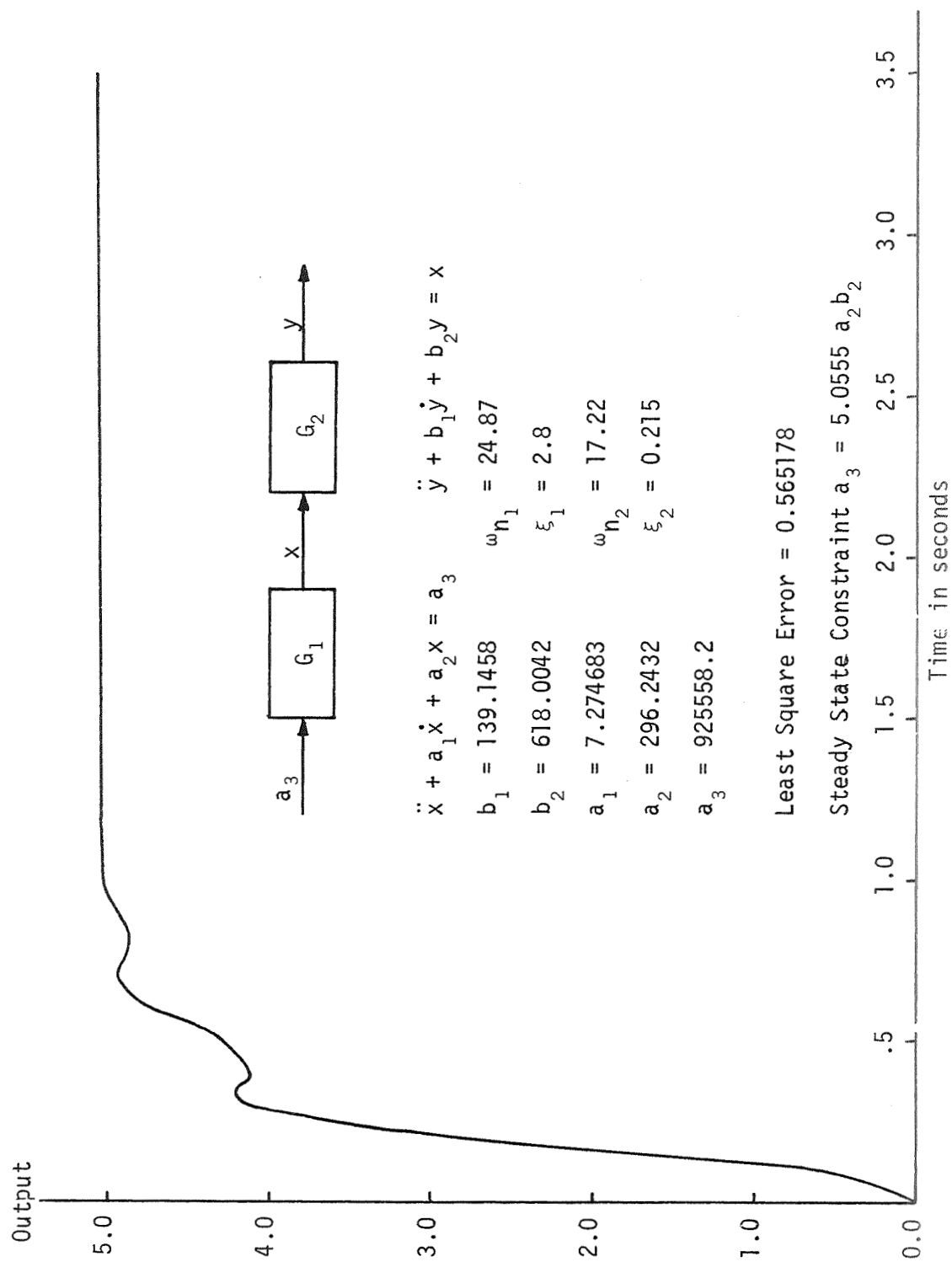
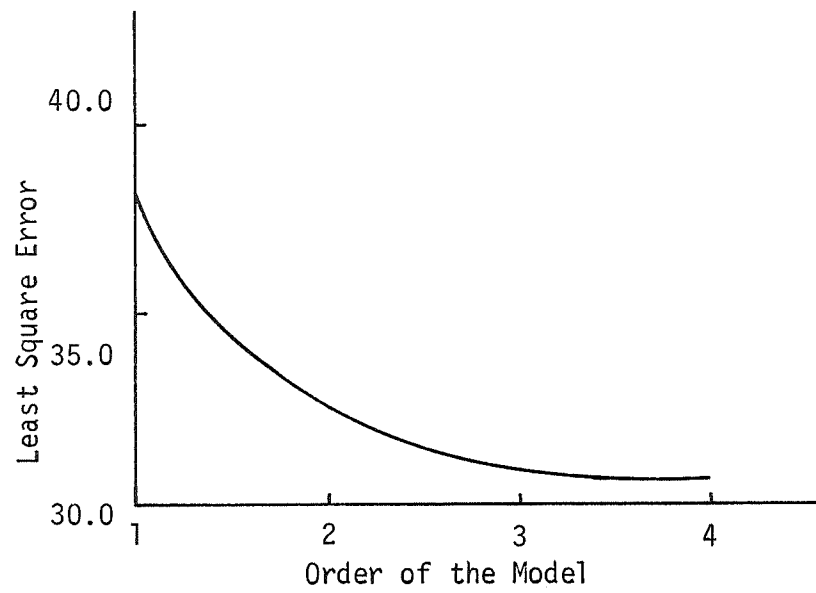
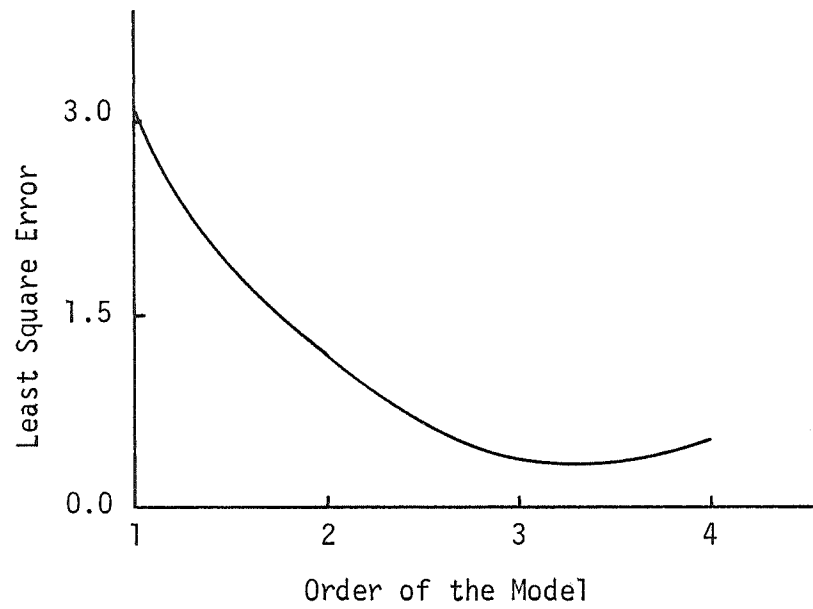


Figure 13. Fourth Order Model of Sinusoidal Data Shown in Figure 5



(a) Random Input



(b) Sinusoidal Input

Figure 14. Least Square Error vs. Order of the Models

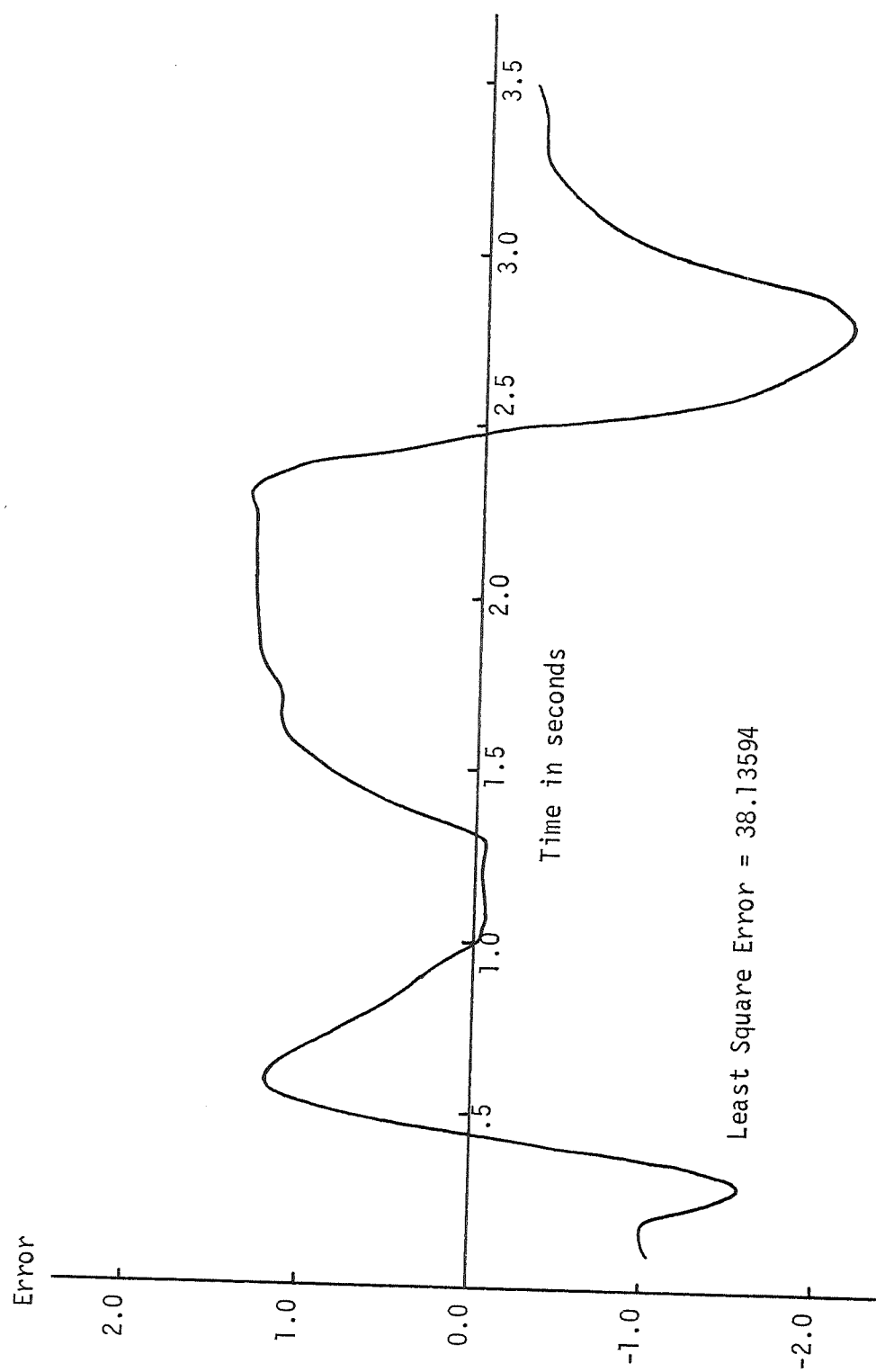


Figure 15. Error in First Order Model of Random Data Shown in Figure 5

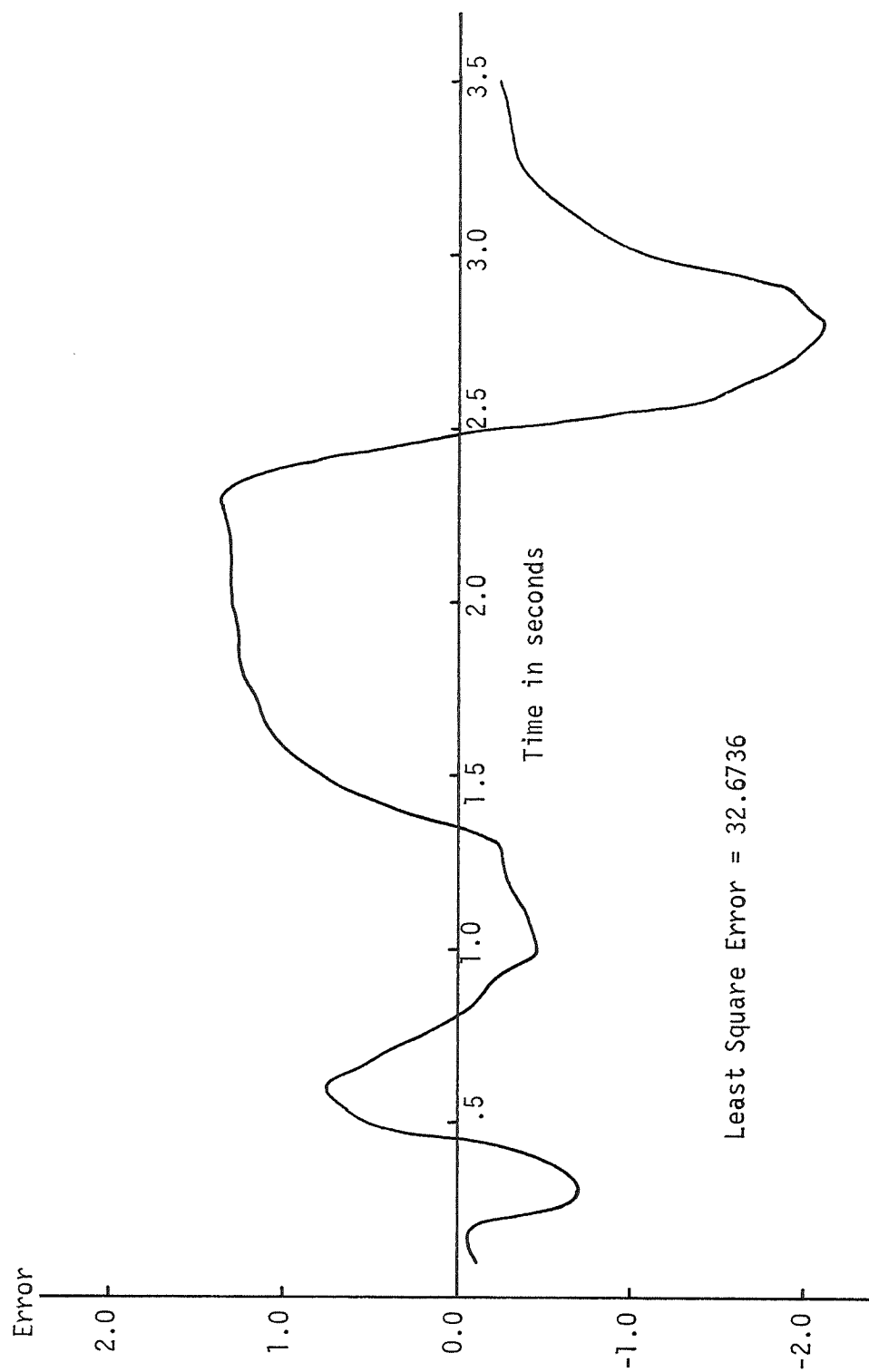


Figure 16. Error in Second Order Model of Random Data Shown in Figure 5

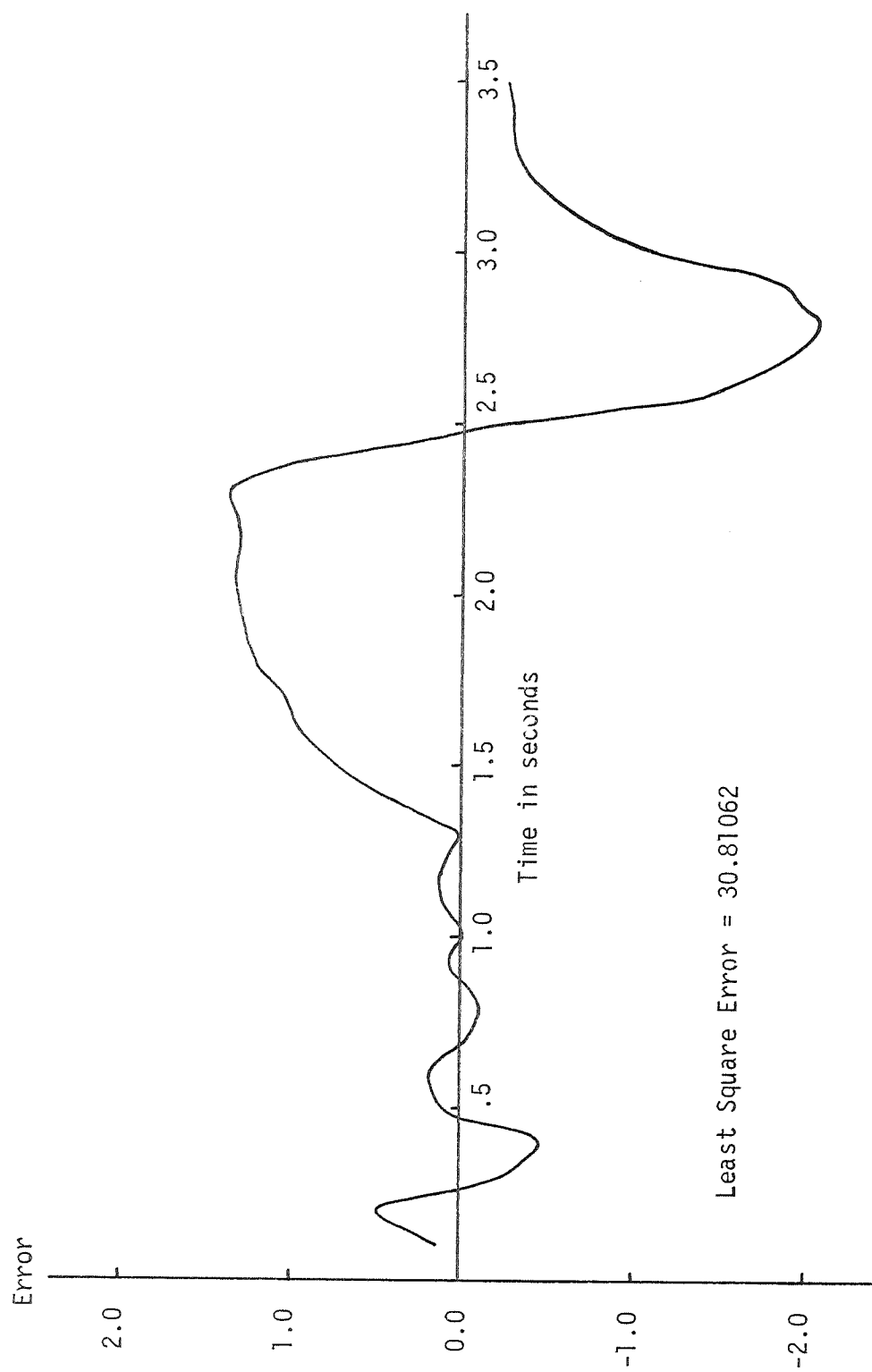


Figure 17. Error in Third Order Model of Random Data Shown in Figure 5

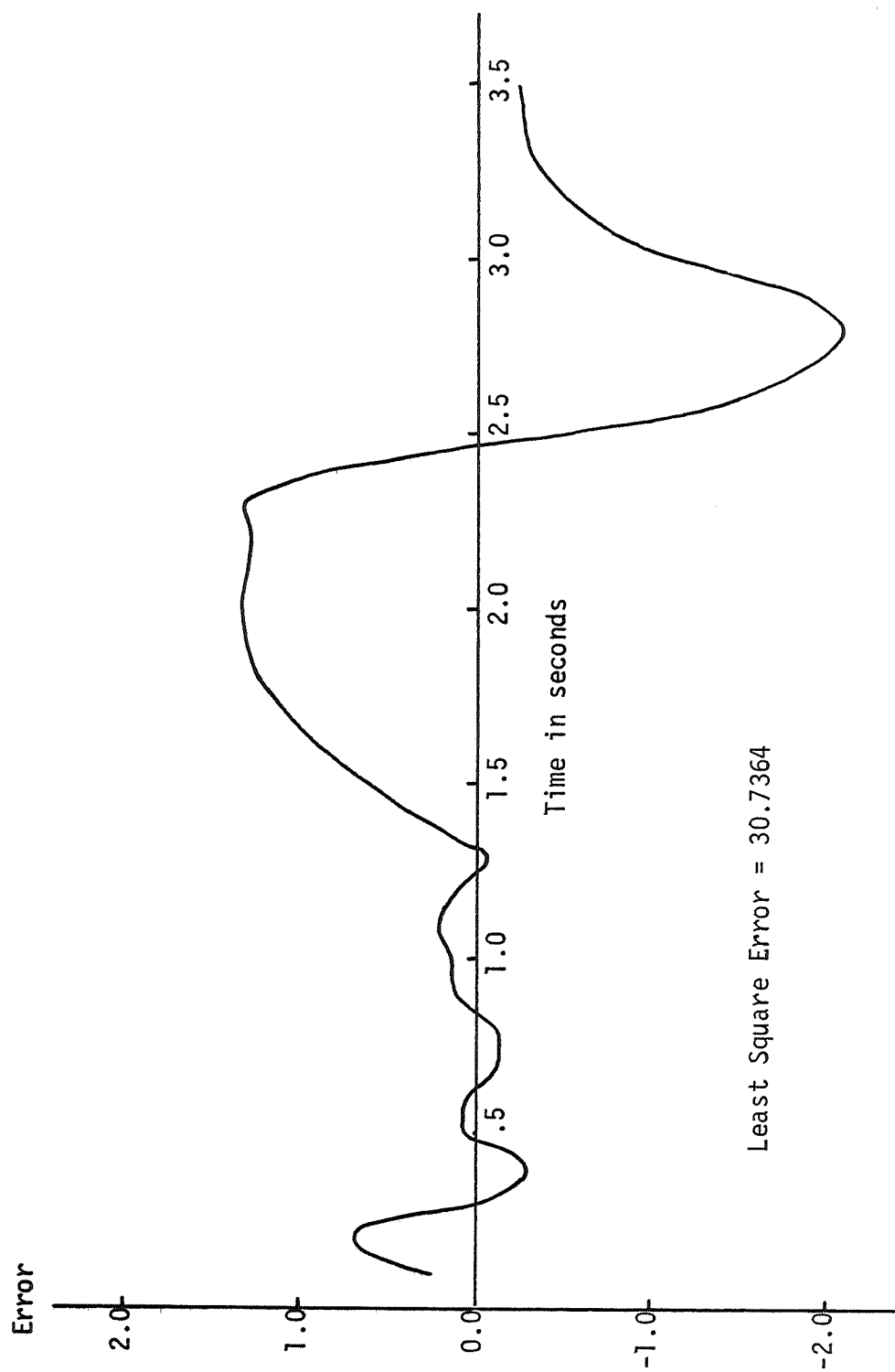


Figure 18. Error in Fourth Order Model of Random Data Shown in Figure 5

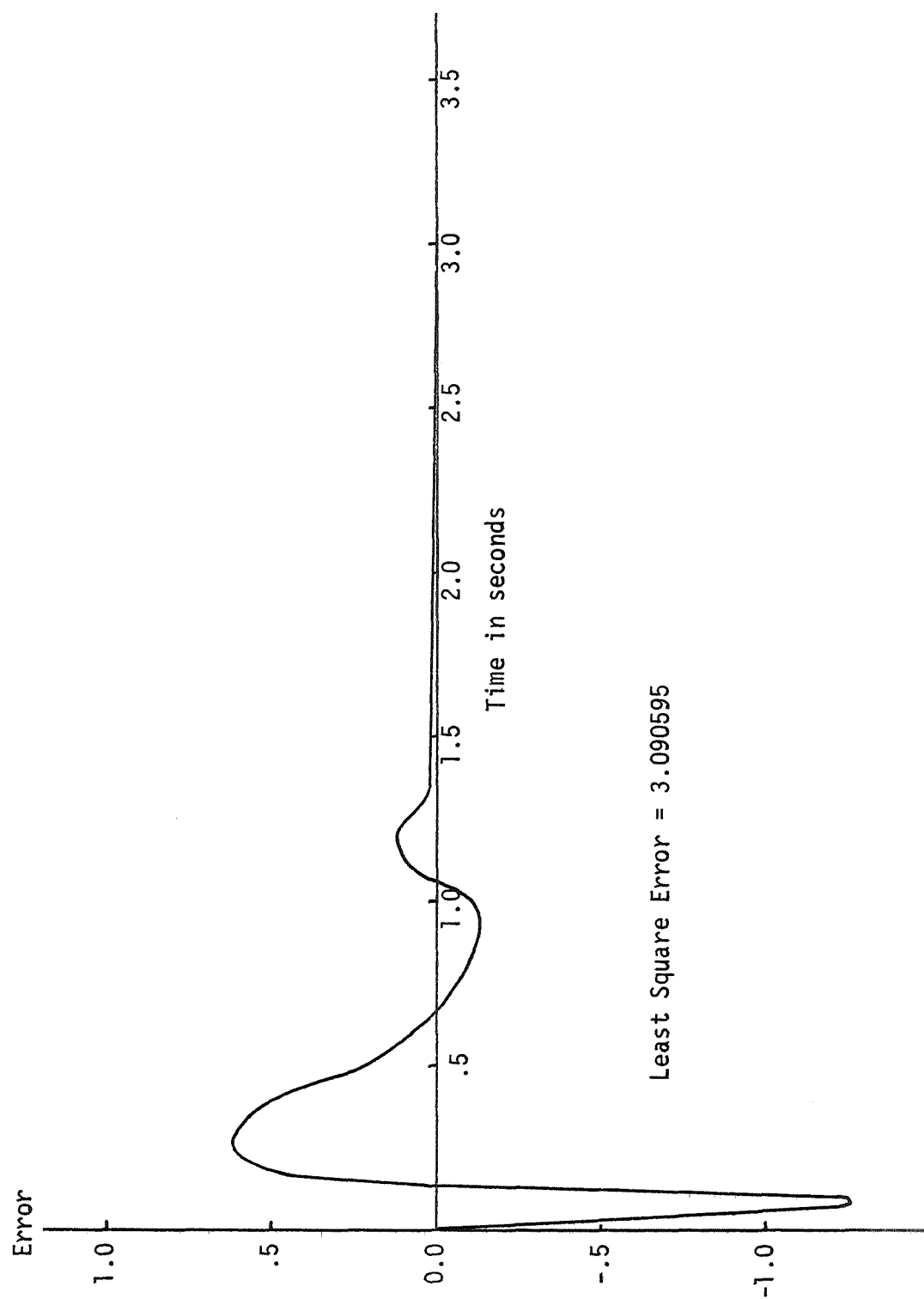


Figure 19. Error in First Order Model of Sinusoidal Data Shown in Figure 5

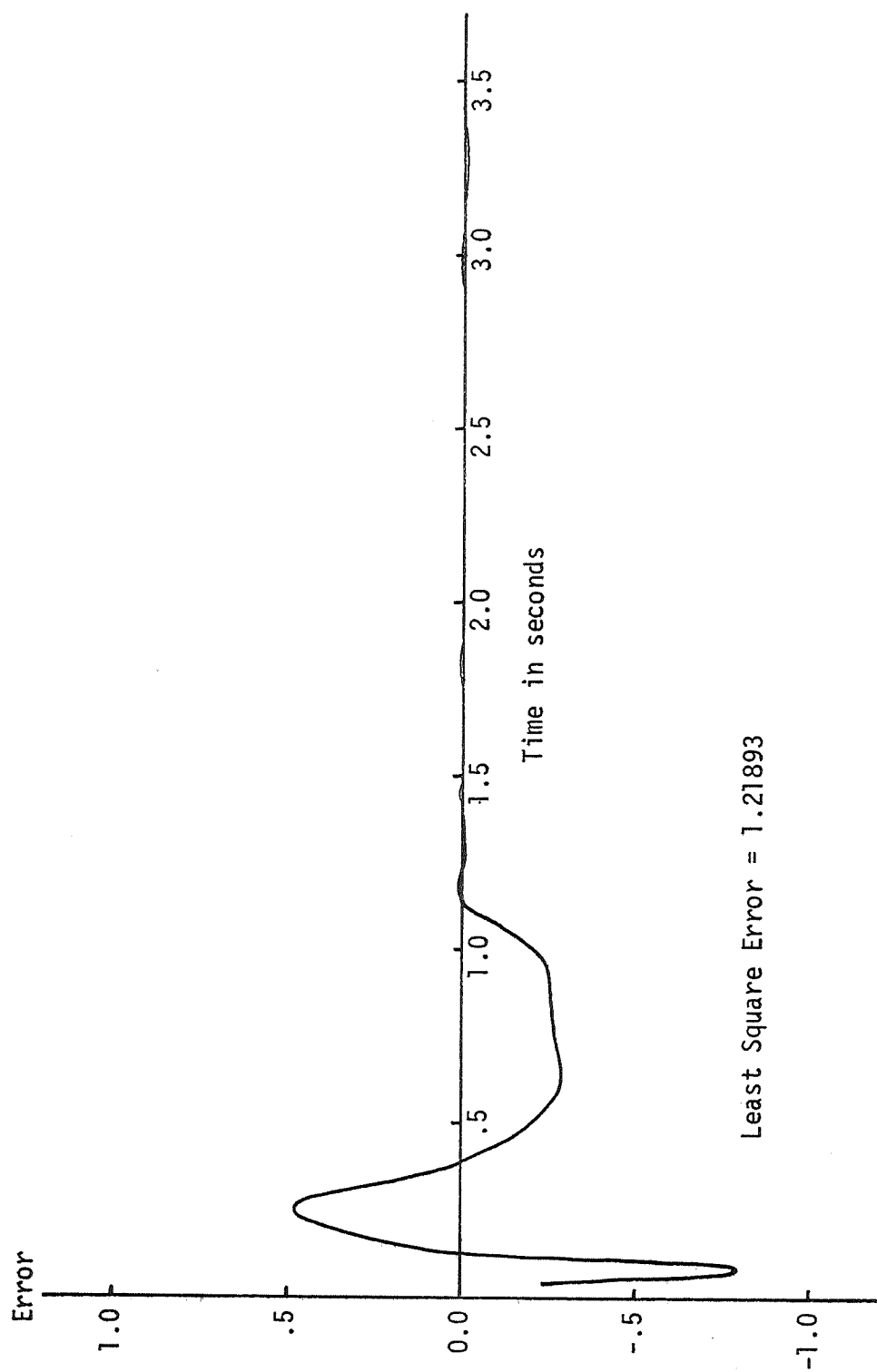


Figure 20. Error in Second Order Model of Sinusoidal Data Shown in Figure 5

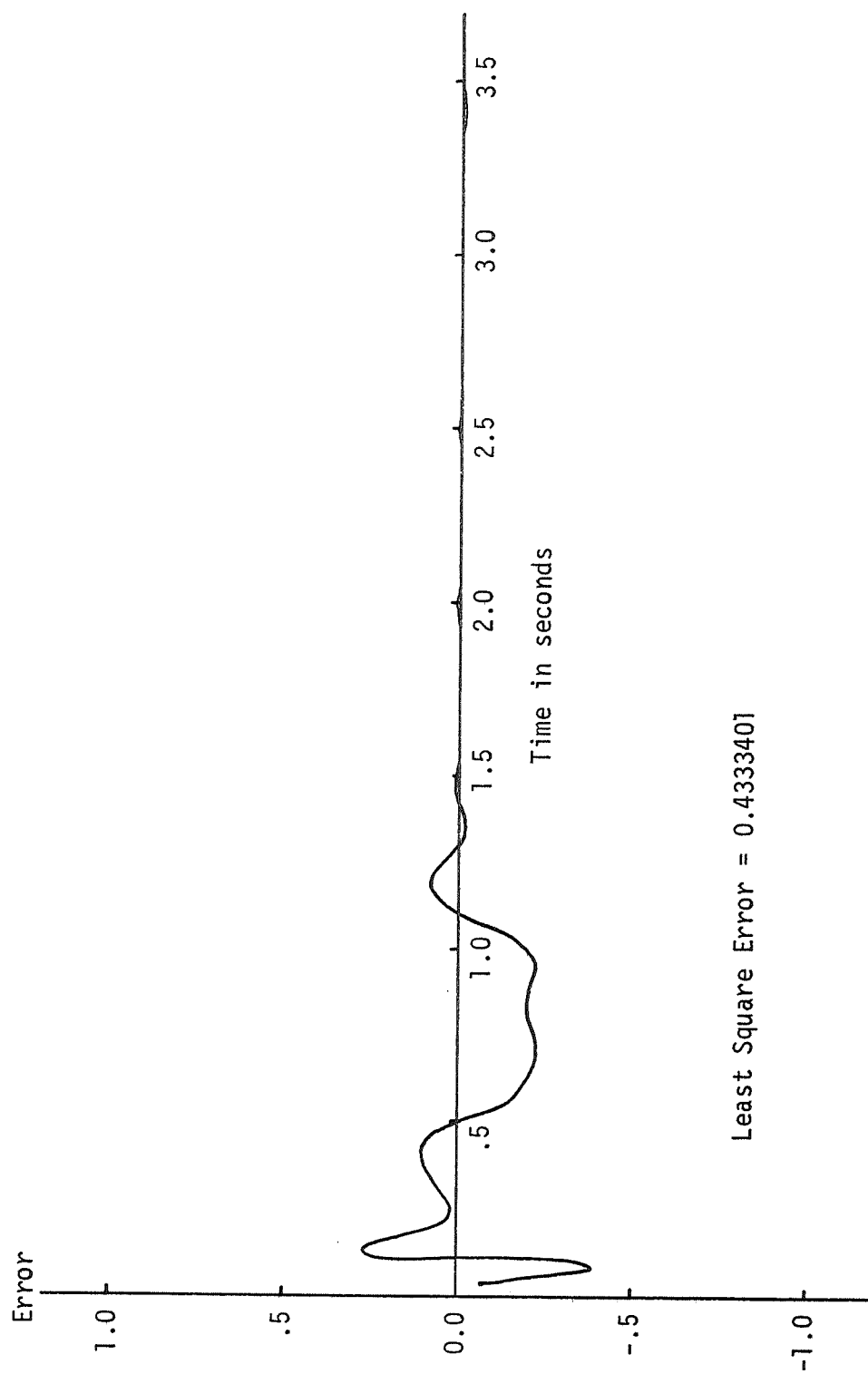


Figure 21. Error in Third Order Model of Sinusoidal Data Shown in Figure 5

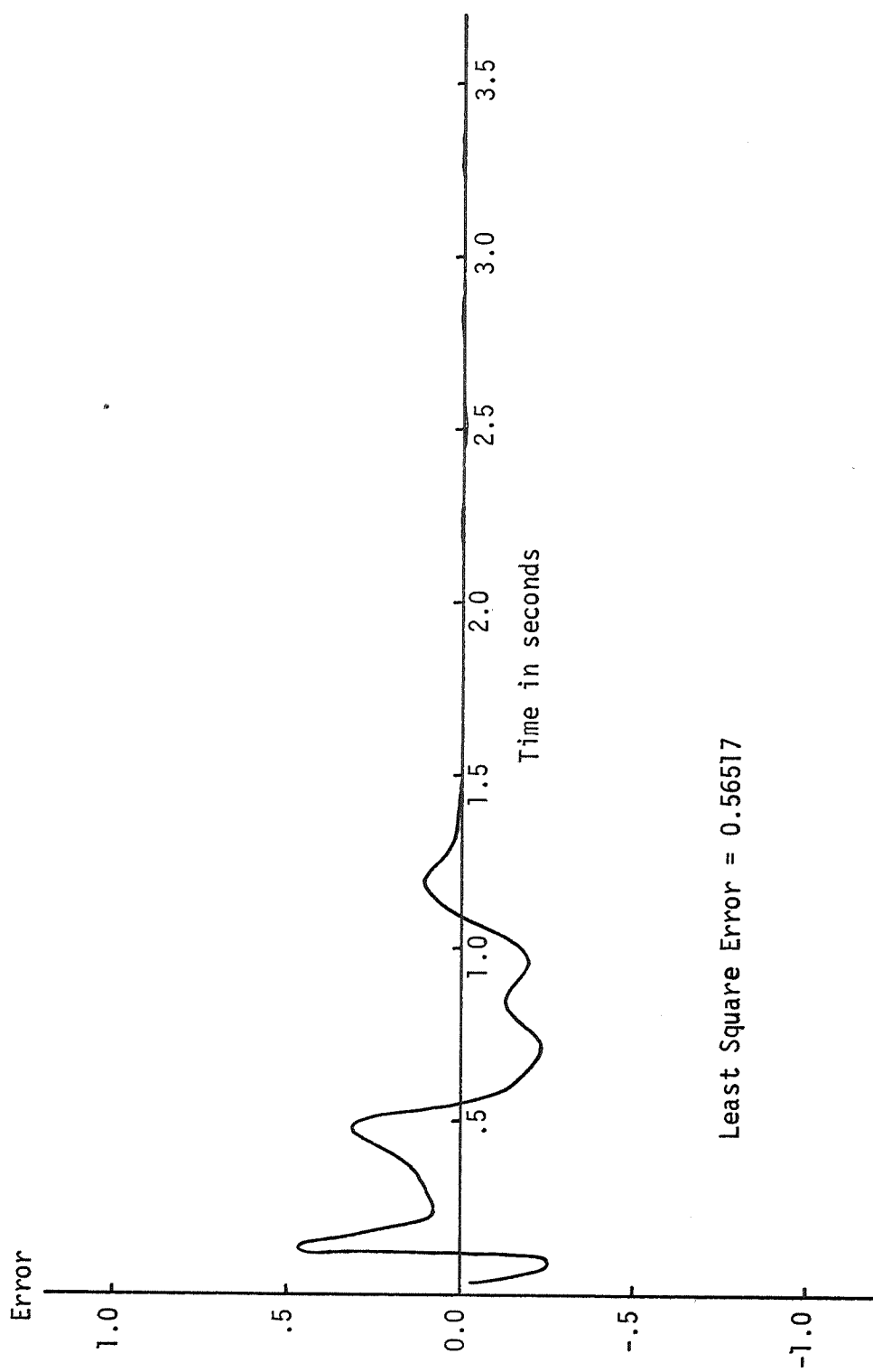


Figure 22. Error in Fourth Order Model of Sinusoidal Data Shown in Figure 5